Microwave Devices and Circuits

Third Edition

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Chapter 3

Microwave Transmission Lines

3-0 INTRODUCTION

Conventional two-conductor transmission lines are commonly used for transmitting microwave energy. If a line is properly matched to its characteristic impedance at each terminal, its efficiency can reach a maximum.

In ordinary circuit theory it is assumed that all impedance elements are lumped constants. This is not true for a long transmission line over a wide range of frequencies. Frequencies of operation are so high that inductances of short lengths of conductors and capacitances between short conductors and their surroundings cannot be neglected. These inductances and capacitances are distributed along the length of a conductor, and their effects combine at each point of the conductor. Since the wavelength is short in comparison to the physical length of the line, distributed parameters cannot be represented accurately by means of a lumped-parameter equivalent circuit. Thus microwave transmission lines can be analyzed in terms of voltage, current, and impedance only by the distributed-circuit theory. If the spacing between the lines is smaller than the wavelength of the transmitted signal, the transmission line must be analyzed as a waveguide.

3-1 TRANSMISSION-LINE EQUATIONS AND SOLUTIONS

3-1-1 Transmission-Line Equations

A transmission line can be analyzed either by the solution of Maxwell's field equations or by the methods of distributed-circuit theory. The solution of Maxwell's equations involves three space variables in addition to the time variable. The distributed-circuit method, however, involves only one space variable in addition to

the time variable. In this section the latter method is used to analyze a transmission line in terms of the voltage, current, impedance, and power along the line.

Based on uniformly distributed-circuit theory, the schematic circuit of a conventional two-conductor transmission line with constant parameters R, L, G, and C is shown in Fig. 3-1-1. The parameters are expressed in their respective names per unit length, and the wave propagation is assumed in the positive z direction.

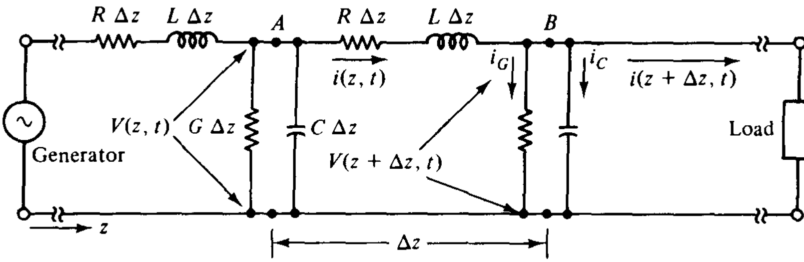


Figure 3-1-1 Elementary section of a transmission line.

By Kirchhoff's voltage law, the summation of the voltage drops around the central loop is given by

$$v(z, t) = i(z, t)R\,\Delta z + L\Delta z\frac{\partial i(z, t)}{\partial t} + v(z, t) + \frac{\partial v(z, t)}{dz}\,\Delta z \tag{3-1-1}$$

Rearranging this equation, dividing it by Δz, and then omitting the argument (z, t), which is understood, we obtain

$$-\frac{\partial v}{\partial z} = Ri + L\frac{\partial i}{\partial t} \tag{3-1-2}$$

Using Kirchhoff's current law, the summation of the currents at point B in Fig. 3-1-1 can be expressed as

$$\begin{aligned} i(z, t) &= v(z + \Delta z, t)G\,\Delta z + C\,\Delta z\,\frac{\partial v(z + \Delta z, t)}{\partial t} + i(z + \Delta z, t) \\ &= \left[v(z, t) + \frac{\partial v(z, t)}{\partial z}\,\Delta z\right]G\,\Delta z \\ &\quad + C\,\Delta z\,\frac{\partial}{\partial t}\left[v(z, t) + \frac{\partial v(z, t)}{\partial z}\,\Delta z\right] + i(z, t) + \frac{\partial i(z, t)}{\partial z}\,\Delta z \end{aligned} \tag{3-1-3}$$

By rearranging the preceding equation, dividing it by Δz, omitting (z, t), and assuming Δz equal to zero, we have

$$-\frac{\partial i}{\partial z} = Gv + C\frac{\partial v}{\partial t} \tag{3-1-4}$$

Then by differentiating Eq. (3-1-2) with respect to z and Eq. (3-1-4) with respect to t and combining the results, the final transmission-line equation in voltage form is

found to be

$$\frac{\partial^2 v}{\partial z^2} = RGv + (RC + LG)\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2} \tag{3-1-5}$$

Also, by differentiating Eq. (3-1-2) with respect to t and Eq. (3-1-4) with respect to z and combining the results, the final transmission-line equation in current form is

$$\frac{\partial^2 i}{\partial z^2} = RGi + (RC + LG)\frac{\partial i}{\partial t} + LC\frac{\partial^2 i}{\partial t^2} \tag{3-1-6}$$

All these transmission-line equations are applicable to the general transient solution. The voltage and current on the line are the functions of both position z and time t. The instantaneous line voltage and current can be expressed as

$$v(z, t) = \text{Re}\, \mathbf{V}(z)e^{j\omega t} \tag{3-1-7}$$

$$i(z, t) = \text{Re}\, \mathbf{I}(z)e^{j\omega t} \tag{3-1-8}$$

where Re stands for "real part of." The factors $\mathbf{V}(z)$ aand $\mathbf{I}(z)$ are complex quantities of the sinusoidal functions of position z on the line and are known as *phasors*. The phasors give the magnitudes and phases of the sinusoidal function at each position of z, and they can be expressed as

$$\mathbf{V}(z) = \mathbf{V}_+ e^{-\gamma z} + \mathbf{V}_- e^{\gamma z} \tag{3-1-9}$$

$$\mathbf{I}(z) = \mathbf{I}_+ e^{-\gamma z} + \mathbf{I}_- e^{\gamma z} \tag{3-1-10}$$

$$\gamma = \alpha + j\beta \qquad \text{(propagation constant)} \tag{3-1-11}$$

where $\mathbf{V}_+$ and $\mathbf{I}_+$ indicate complex amplitudes in the positive z direction, $\mathbf{V}_-$ and $\mathbf{I}_-$ signify complex amplitudes in the negative z direction, α is the attenuation constant in nepers per unit length, and β is the phase constant in radians per unit length.

If we substitute $j\omega$ for $\partial/\partial t$ in Eqs. (3-1-2), (3-1-4), (3-1-5), and (3-1-6) and divide each equation by $e^{j\omega t}$, the transmission-line equations in phasor form of the frequency domain become

$$\frac{d\mathbf{V}}{dz} = -\mathbf{ZI} \tag{3-1-12}$$

$$\frac{d\mathbf{I}}{dz} = -\mathbf{YV} \tag{3-1-13}$$

$$\frac{d^2\mathbf{V}}{dz^2} = \gamma^2 \mathbf{V} \tag{3-1-14}$$

$$\frac{d^2\mathbf{I}}{dz^2} = \gamma^2 \mathbf{I} \tag{3-1-15}$$

in which the following substitutions have been made:

$$\mathbf{Z} = R + j\omega L \qquad \text{(ohms per unit length)} \tag{3-1-16}$$

$$\mathbf{Y} = G + j\omega C \qquad \text{(mhos per unit length)} \tag{3-1-17}$$

$$\gamma = \sqrt{ZY} = \alpha + j\beta \qquad \text{(propagation constant)} \tag{3-1-18}$$

For a lossless line, $R = G = 0$, and the transmission-line equations are expressed as

$$\frac{d\mathbf{V}}{dz} = -j\omega L\mathbf{I} \tag{3-1-19}$$

$$\frac{d\mathbf{I}}{dz} = -j\omega C\mathbf{V} \tag{3-1-20}$$

$$\frac{d^2\mathbf{V}}{dz^2} = -\omega^2 LC\mathbf{V} \tag{3-1-21}$$

$$\frac{d^2\mathbf{I}}{dz^2} = -\omega^2 LC\mathbf{I} \tag{3-1-22}$$

It is interesting to note that Eqs. (3-1-14) and (3-1-15) for a transmission line are similar to equations of the electric and magnetic waves, respectively. The only difference is that the transmission-line equations are one-dimensional.

3-1-2 Solutions of Transmission-Line Equations

The one possible solution for Eq. (3-1-14) is

$$\mathbf{V} = \mathbf{V}_+ e^{-\gamma z} + \mathbf{V}_- e^{\gamma z} = \mathbf{V}_+ e^{-\alpha z} e^{-j\beta z} + \mathbf{V}_- e^{\alpha z} e^{j\beta z} \tag{3-1-23}$$

The factors $\mathbf{V}_+$ and $\mathbf{V}_-$ represents complex quantities. The term involving $e^{-j\beta z}$ shows a wave traveling in the positive z direction, and the term with the factor $e^{j\beta z}$ is a wave going in the negative z direction. The quantity βz is called the *electrical length of the line* and is measured in radians.

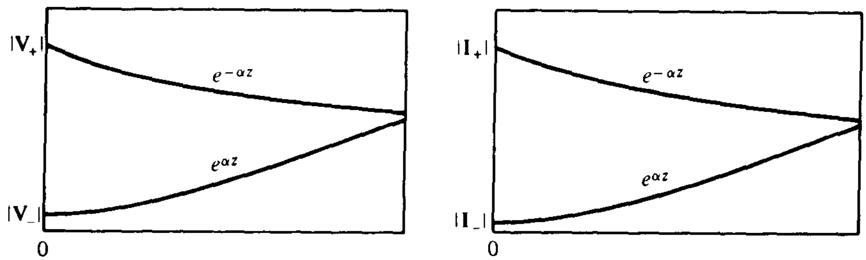
Similarly, the one possible solution for Eq. (3-1-15) is

$$\mathbf{I} = \mathbf{Y}_0(\mathbf{V}_+ e^{-\gamma z} - \mathbf{V}_- e^{\gamma z}) = \mathbf{Y}_0(\mathbf{V}_+ e^{-\alpha z} e^{-j\beta z} - \mathbf{V}_- e^{\alpha z} e^{j\beta z}) \tag{3-1-24}$$

In Eq. (3-1-24) the characteristic impedance of the line is defined as

$$\mathbf{Z}_0 = \frac{1}{\mathbf{Y}_0} \equiv \sqrt{\frac{\mathbf{Z}}{\mathbf{Y}}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = R_0 \pm jX_0 \tag{3-1-25}$$

The magnitude of both voltage and current waves on the line is shown in Fig. 3-1-2.



(a) Voltage wave (b) Current wave

Figure 3-1-2 Magnitude of voltage and current traveling waves.

At microwave frequencies it can be seen that

$$R \ll \omega L \quad \text{and} \quad G \ll \omega C \tag{3-1-26}$$

By using the binomial expansion, the propagation constant can be expressed as

$$\begin{aligned}
\gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\
&= \sqrt{(j\omega)^2 LC}\sqrt{\left(1 + \frac{R}{j\omega L}\right)\left(1 + \frac{G}{j\omega C}\right)} \\
&\approx j\omega\sqrt{LC}\left[\left(1 + \frac{1}{2}\frac{R}{j\omega L}\right)\left(1 + \frac{1}{2}\frac{G}{j\omega C}\right)\right] \\
&\approx j\omega\sqrt{LC}\left[1 + \frac{1}{2}\left(\frac{R}{j\omega L} + \frac{G}{j\omega C}\right)\right] \\
&= \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) + j\omega\sqrt{LC}
\end{aligned} \tag{3-1-27}$$

Therefore the attenuation and phase constants are, respectively, given by

$$\alpha = \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right) \tag{3-1-28}$$

$$\beta = \omega\sqrt{LC} \tag{3-1-29}$$

Similarly, the characteristic impedance is found to be

$$\begin{aligned}
\mathbf{Z}_0 &= \sqrt{\frac{R + j\omega L}{G + j\omega C}} \\
&= \sqrt{\frac{L}{C}}\left(1 + \frac{R}{j\omega L}\right)^{1/2}\left(1 + \frac{G}{j\omega C}\right)^{-1/2} \\
&\approx \sqrt{\frac{L}{C}}\left(1 + \frac{1}{2}\frac{R}{j\omega L}\right)\left(1 - \frac{1}{2}\frac{G}{j\omega C}\right) \\
&\approx \sqrt{\frac{L}{C}}\left[1 + \frac{1}{2}\left(\frac{R}{j\omega L} - \frac{G}{j\omega C}\right)\right] \\
&\approx \sqrt{\frac{L}{C}}
\end{aligned} \tag{3-1-30}$$

From Eq. (3-1-29) the phase velocity is

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \tag{3-1-31}$$

The product of LC is independent of the size and separation of the conductors and depends only on the permeability μ and permittivity of ϵ of the insulating medium. If a lossless transmission line used for microwave frequencies has an air dielectric and contains no ferromagnetic materials, free-space parameters can be assumed.

Thus the numerical value of $1/\sqrt{LC}$ for air-insulated conductors is approximately equal to the velocity of light in vacuum. That is,

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c = 3 \times 10^8 \text{ m/s} \tag{3-1-32}$$

When the dielectric of a lossy microwave transmission line is not air, the phase velocity is smaller than the velocity of light in vacuum and is given by

$$v_\epsilon = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{\mu_r \epsilon_r}} \tag{3-1-33}$$

In general, the relative phase velocity factory can be defined as

$$\text{Velocity factor} = \frac{\text{actual phase velocity}}{\text{velocity of light in vacuum}}$$

$$v_r = \frac{v_\epsilon}{c} = \frac{1}{\sqrt{\mu_r \epsilon_r}} \tag{3-1-34}$$

A low-loss transmission line filled only with dielectric medium, such as a coaxial line with solid dielectric between conductors, has a velocity factor on the order of about 0.65.

Example 3-1-1: Line Characteristic Impedance and Propagation Constant

A transmission line has the following parameters:

$$R = 2\ \Omega/\text{m} \qquad G = 0.5 \text{ mmho/m} \qquad f = 1 \text{ GHz}$$

$$L = 8 \text{ nH/m} \qquad C = 0.23 \text{ pF}$$

Calculate: **(a)** the characteristic impedance; **(b)** the propagation constant.

Solution

a. From Eq. (3-1-25) the line characteristic impedance is

$$\mathbf{Z}_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{2 + j2\pi \times 10^9 \times 8 \times 10^{-9}}{0.5 \times 10^{-3} + j2\pi \times 10^9 \times 0.23 \times 10^{-12}}}$$

$$= \sqrt{\frac{50.31\underline{/87.72^\circ}}{15.29 \times 10^{-4}\underline{/70.91^\circ}}} = 181.39\underline{/8.40^\circ} = 179.44 + j26.50$$

b. From Eq. (3-1-18) the propagation constant is

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(50.31\underline{/87.72^\circ})(15.29 \times 10^{-4}\underline{/70.91^\circ})}$$

$$= \sqrt{769.24 \times 10^{-4}\underline{/158.63^\circ}}$$

$$= 0.2774\underline{/79.31^\circ} = 0.051 + j0.273$$

3-2 REFLECTION COEFFICIENT AND TRANSMISSION COEFFICIENT

3-2-1 Reflection Coefficient

In the analysis of the solutions of transmission-line equations in Section 3-1, the traveling wave along the line contains two components: one traveling in the positive z direction and the other traveling the negative z direction. If the load impedance is equal to the line characteristic impedance, however, the reflected traveling wave does not exist.

Figure 3-2-1 shows a transmission line terminated in an impedance $\mathbf{Z}_\ell$. It is usually more convenient to start solving the transmission-line problem from the receiving rather than the sending end, since the voltage-to-current relationship at the load point is fixed by the load impedance. The incident voltage and current waves traveling along the transmission line are given by

$$\mathbf{V} = \mathbf{V}_+ e^{-\gamma z} + \mathbf{V}_- e^{+\gamma z} \tag{3-2-1}$$

$$\mathbf{I} = \mathbf{I}_+ e^{-\gamma z} + \mathbf{I}_- e^{+\gamma z} \tag{3-2-2}$$

in which the current wave can be expressed in terms of the voltage by

$$\mathbf{I} = \frac{\mathbf{V}_+}{\mathbf{Z}_0} e^{-\gamma z} - \frac{\mathbf{V}^-}{\mathbf{Z}_0} e^{\gamma z} \tag{3-2-3}$$

If the line has a length of ℓ, the voltage and current at the receiving end become

$$\mathbf{V}_\ell = \mathbf{V}_+ e^{-\gamma\ell} + \mathbf{V}_- e^{\gamma\ell} \tag{3-2-4}$$

$$\mathbf{I}_\ell = \frac{1}{\mathbf{Z}_0}(\mathbf{V}_+ e^{-\gamma\ell} - \mathbf{V}_- e^{\gamma\ell}) \tag{3-2-5}$$

The ratio of the voltage to the current at the receiving end is the load impedance. That is,

$$\mathbf{Z}_\ell = \frac{\mathbf{V}_\ell}{\mathbf{I}_\ell} = \mathbf{Z}_0 \frac{\mathbf{V}_+ e^{-\gamma\ell} + \mathbf{V}_- e^{\gamma\ell}}{\mathbf{V}_+ e^{-\gamma\ell} - \mathbf{V}_- e^{\gamma\ell}} \tag{3-2-6}$$

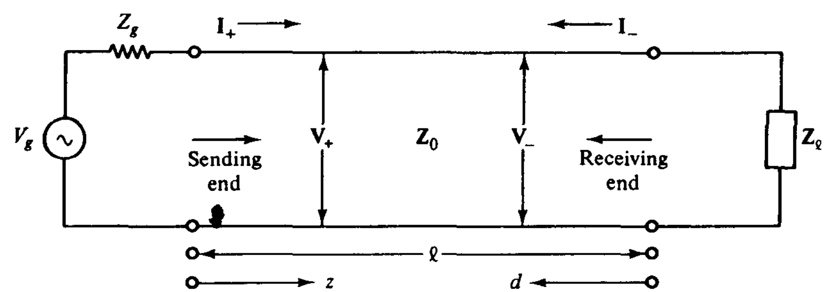


Figure 3-2-1 Transmission line terminated in a load impedance.

The reflection coefficient, which is designated by $\mathbf{\Gamma}$ (gamma), is defined as

$$\text{Reflection coefficient} \equiv \frac{\text{reflected voltage or current}}{\text{incident voltage or current}}$$

$$\mathbf{\Gamma} \equiv \frac{\mathbf{V}_{\text{ref}}}{\mathbf{V}_{\text{inc}}} = \frac{-\mathbf{I}_{\text{ref}}}{\mathbf{I}_{\text{inc}}} \tag{3-2-7}$$

If Eq. (3-2-6) is solved for the ratio of the reflected voltage at the receiving end, which is $\mathbf{V}_- e^{\gamma\ell}$, to the incident voltage at the receiving end, which is $\mathbf{V}_+ e^{\gamma\ell}$, the result is the reflection coefficient at the receiving end:

$$\mathbf{\Gamma}_\ell = \frac{\mathbf{V}_- e^{\gamma\ell}}{\mathbf{V}_+ e^{-\gamma\ell}} = \frac{\mathbf{Z}_\ell - \mathbf{Z}_0}{\mathbf{Z}_\ell + \mathbf{Z}_0} \tag{3-2-8}$$

If the load impedance and/or the characteristic impedance are complex quantities, as is usually the case, the reflection coefficient is generally a complex quantity that can be expressed as

$$\mathbf{\Gamma}_\ell = |\mathbf{\Gamma}_\ell| e^{j\theta_\ell} \tag{3-2-9}$$

where $|\mathbf{\Gamma}_\ell|$ is the magnitude and never greater than unity—that is, $|\mathbf{\Gamma}_\ell| \leq 1$. Note that θ_ℓ is the phase angle between the incident and reflected voltages at the receiving end. It is usually called the phase angle of the reflection coefficient.

The general solution of the reflection coefficient at any point on the line, then, corresponds to the incident and reflected waves at that point, each attenuated in the direction of its own progress along the line. The generalized reflection coefficient is defined as

$$\mathbf{\Gamma} \equiv \frac{\mathbf{V}_- e^{\gamma z}}{\mathbf{V}_+ e^{-\gamma z}} \tag{3-2-10}$$

From Fig. 3-2-1 let $z = \ell - d$. Then the reflection coefficient at some point located a distance d from the receiving end is

$$\mathbf{\Gamma}_d = \frac{\mathbf{V}_- e^{\gamma(\ell - d)}}{\mathbf{V}_+ e^{-\gamma(\ell - d)}} = \frac{\mathbf{V}_- e^{\gamma\ell}}{\mathbf{V}_+ e^{-\gamma\ell}} e^{-2\gamma d} = \mathbf{\Gamma}_\ell e^{-2\gamma d} \tag{3-2-11}$$

Next, the reflection coefficient at that point can be expressed in terms of the reflection coefficient at the receiving end as

$$\mathbf{\Gamma}_d = \mathbf{\Gamma}_\ell e^{-2\alpha d} e^{-j2\beta d} = |\mathbf{\Gamma}_\ell| e^{-2\alpha d} e^{j(\theta_\ell - 2\beta d)} \tag{3-2-12}$$

This is a very useful equation for determining the reflection coefficient at any point along the line. For a lossy line, both the magnitude and phase of the reflection coefficient are changing in an inward-spiral way as shown in Fig. 3-2-2. For a lossless line, $\alpha = 0$, the magnitude of the reflection coefficient remains constant, and only the phase of $\mathbf{\Gamma}$ is changing circularly toward the generator with an angle of $-2\beta d$ as shown in Fig. 3-2-3.

It is evident that $\mathbf{\Gamma}_\ell$ will be zero and there will be no reflection from the receiving end when the terminating impedance is equal to the characteristic impedance

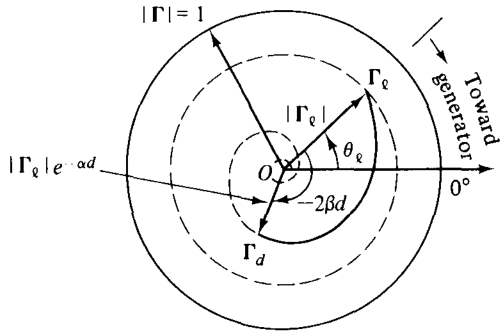


Figure 3-2-2 Reflection coefficient for lossy line.

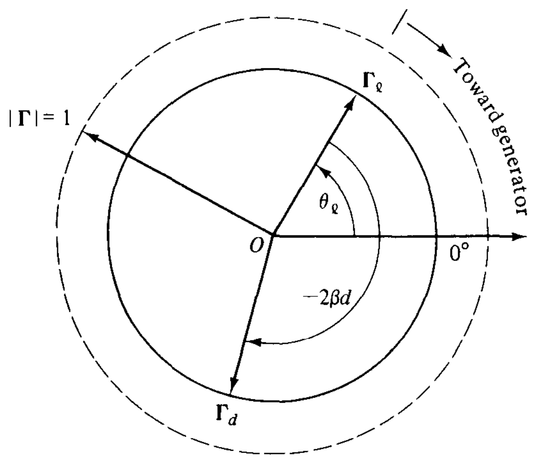


Figure 3-2-3 Reflection coefficient for lossless line.

of the line. Thus a terminating impedance that differs from the characteristic impedance will create a reflected wave traveling toward the source from the termination. The reflection, upon reaching the sending end, will itself be reflected if the source impedance is different from the line characteristic impedance at the sending end.

3-2-2 Transmission Coefficient

A transmission line terminated in its characteristic impedance $\mathbf{Z}_0$ is called a *properly terminated* line. Otherwise it is called an *improperly terminated* line. As described earlier, there is a reflection coefficient $\mathbf{\Gamma}$ at any point along an improperly terminated line. According to the principle of conservation of energy, the incident power minus the reflected power must be equal to the power transmitted to the load. This can be expressed as

$$1 - \mathbf{\Gamma}_\ell^2 = \frac{\mathbf{Z}_0}{\mathbf{Z}_\ell} \mathbf{T}^2 \tag{3-2-13}$$

Equation (3-2-13) will be verified later. The letter $\mathbf{T}$ represents the transmission coefficient, which is defined as

$$\mathbf{T} \equiv \frac{\text{transmitted voltage or current}}{\text{incident voltage or current}} = \frac{\mathbf{V}_{\text{tr}}}{\mathbf{V}_{\text{inc}}} = \frac{\mathbf{I}_{\text{tr}}}{\mathbf{I}_{\text{inc}}} \tag{3-2-14}$$

Figure 3-2-4 shows the transmission of power along a transmission line where P_{inc} is the incident power, P_{ref} the reflected power, and P_{tr} the transmitted power.

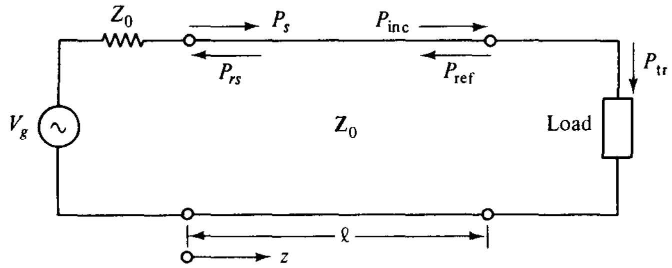


Figure 3-2-4 Power transmission on a line.

Let the traveling waves at the receiving end be

$$\mathbf{V}_+ e^{-\gamma\ell} + \mathbf{V}_- e^{\gamma\ell} = \mathbf{V}_{\text{tr}} e^{-\gamma\ell} \tag{3-2-15}$$

$$\frac{\mathbf{V}_+}{\mathbf{Z}_0} e^{-\gamma\ell} - \frac{\mathbf{V}_-}{\mathbf{Z}_0} e^{\gamma\ell} = \frac{\mathbf{V}_{\text{tr}}}{\mathbf{Z}_\ell} e^{-\gamma\ell} \tag{3-2-16}$$

Multiplication of Eq. (3-2-16) by $\mathbf{Z}_\ell$ and substitution of the result in Eq. (3-2-15) yield

$$\mathbf{\Gamma}_\ell = \frac{\mathbf{V}_- e^{\gamma\ell}}{\mathbf{V}_+ e^{-\gamma\ell}} = \frac{\mathbf{Z}_\ell - \mathbf{Z}_0}{\mathbf{Z}_\ell + \mathbf{Z}_0} \tag{3-2-17}$$

which, in turn, on substitution back into Eq. (3-2-15), results in

$$\mathbf{T} = \frac{\mathbf{V}_{\text{tr}}}{\mathbf{V}_+} = \frac{2\mathbf{Z}_\ell}{\mathbf{Z}_\ell + \mathbf{Z}_0} \tag{3-2-18}$$

The power carried by the two waves in the side of the incident and reflected waves is

$$P_{\text{inr}} = P_{\text{inc}} - P_{\text{ref}} = \frac{(\mathbf{V}_+ e^{-\alpha\ell})^2}{2\mathbf{Z}_0} - \frac{(\mathbf{V}_- e^{\alpha\ell})^2}{2\mathbf{Z}_0} \tag{3-2-19}$$

The power carried to the load by the transmitted waves is

$$P_{\text{tr}} = \frac{(\mathbf{V}_{\text{tr}} e^{-\alpha\ell})^2}{2\mathbf{Z}_\ell} \tag{3-2-20}$$

By setting $P_{\text{inr}} = P_{\text{tr}}$ and using Eqs. (3-2-17) and (3-2-18), we have

$$\mathbf{T}^2 = \frac{\mathbf{Z}_\ell}{\mathbf{Z}_0}(1 - \mathbf{\Gamma}_\ell^2) \tag{3-2-21}$$

This relation verifies the previous statement that the transmitted power is equal to the difference of the incident power and reflected power.

Example 3-2-1: Reflection Coefficient and Transmission Coefficient

A certain transmission line has a characteristic impedance of $75 + j0.01\ \Omega$ and is terminated in a load impedance of $70 + j50\ \Omega$. Compute **(a)** the reflection coefficient; **(b)** the transmission coefficient. Verify: **(c)** the relationship shown in Eq. (3-2-21); **(d)** the transmission coefficient equals the algebraic sum of 1 plus the reflection coefficient as shown in Eq. (2-3-18).

Solution

a. From Eq. (3-2-17) the reflection coefficient is

$$\mathbf{\Gamma} = \frac{\mathbf{Z}_\ell - \mathbf{Z}_0}{\mathbf{Z}_\ell + \mathbf{Z}_0} = \frac{70 + j50 - (75 + j0.01)}{70 + j50 + (75 + j0.01)}$$

$$= \frac{50.24\underline{/95.71^\circ}}{153.38\underline{/19.03^\circ}} = 0.33\underline{/76.68^\circ} = 0.08 + j0.32$$

b. From Eq. (3-2-18) the transmission coefficient is

$$\mathbf{T} = \frac{2\mathbf{Z}_\ell}{\mathbf{Z}_\ell + \mathbf{Z}_0} = \frac{2(70 + j50)}{70 + j50 + (75 + j0.01)}$$

$$= \frac{172.05\underline{/35.54^\circ}}{153.38\underline{/19.03^\circ}} = 1.12\underline{/16.51^\circ} = 1.08 + j0.32$$

c.

$$\mathbf{T}^2 = (1.12\underline{/16.51^\circ})^2 = 1.25\underline{/33.02^\circ}$$

$$\frac{\mathbf{Z}_\ell}{\mathbf{Z}_0}(1 - \mathbf{\Gamma}^2) = \frac{70 + j50}{75 + j0.01}[1 - (0.33)\underline{/76.68^\circ})^2]$$

$$= \frac{86\underline{/35.54^\circ}}{75\underline{/0^\circ}} \times 1.10\underline{/-2.6^\circ} = 1.25\underline{/33^\circ}$$

Thus Eq. (3-2-21) is verified.

d. From Eq. (2-3-18) we obtain

$$\mathbf{T} = 1.08 + j0.32 = 1 + 0.08 + j0.32 = 1 + \mathbf{\Gamma}$$

3-3 STANDING WAVE AND STANDING-WAVE RATIO

3-3-1 Standing Wave

The general solutions of the transmission-line equation consist of two waves traveling in opposite directions with unequal amplitude as shown in Eqs. (3-1-23) and (3-1-24). Equation (3-1-23) can be written

$$\begin{aligned}\mathbf{V} &= \mathbf{V}_+ e^{-\alpha z} e^{-j\beta z} + \mathbf{V}_- e^{\alpha z} e^{j\beta z} \\ &= \mathbf{V}_+ e^{-\alpha z}[\cos(\beta z) - j\sin(\beta z)] + \mathbf{V}_- e^{\alpha z}[\cos(\beta z) + j\sin(\beta z)] \\ &= (\mathbf{V}_+ e^{-\alpha z} + \mathbf{V}_- e^{\alpha z})\cos(\beta z) - j(\mathbf{V}_+ e^{-\alpha z} - \mathbf{V}_- e^{\alpha z})\sin(\beta z)\end{aligned} \tag{3-3-1}$$

With no loss in generality it can be assumed that $\mathbf{V}_+ e^{-\alpha z}$ and $\mathbf{V}_- e^{\alpha z}$ are real. Then the voltage-wave equation can be expressed as

$$\mathbf{V}_s = \mathbf{V}_0 e^{-j\phi} \tag{3-3-2}$$

This is called the *equation of the voltage standing wave,* where

$$\mathbf{V}_0 = [(\mathbf{V}_+ e^{-\alpha z} + \mathbf{V}_- e^{\alpha z})^2 \cos^2(\beta z) + (\mathbf{V}_+ e^{-\alpha z} - \mathbf{V}_- e^{-\alpha z})^2 \sin^2(\beta z)]^{1/2} \tag{3-3-3}$$

which is called the *standing-wave pattern* of the voltage wave or the amplitude of the standing wave, and

$$\phi = \arctan\left(\frac{\mathbf{V}_+ e^{-\alpha z} - \mathbf{V}_- e^{\alpha z}}{\mathbf{V}_+ e^{-\alpha z} + \mathbf{V}_- e^{\alpha z}} \tan(\beta z)\right) \tag{3-3-4}$$

which is called the *phase pattern of the standing wave*. The maximum and minimum values of Eq. (3-3-3) can be found as usual by differentiating the equation with respect to βz and equating the result to zero. By doing so and substituting the proper values of βz in the equation, we find that

1. The maximum amplitude is

$$\mathbf{V}_{\text{max}} = \mathbf{V}_+ e^{-\alpha z} + \mathbf{V}_- e^{\alpha z} = \mathbf{V}_+ e^{-\alpha z}(1 + |\Gamma|) \tag{3-3-5}$$

   and this occurs at $\beta z = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$.
2. The minimum amplitude is

$$\mathbf{V}_{\text{min}} = \mathbf{V}_+ e^{-\alpha z} - V_- e^{\alpha z} = \mathbf{V}_+ e^{-\alpha z}(1 - |\Gamma|) \tag{3-3-6}$$

   and this occurs at $\beta z = (2n - 1)\pi/2$, where $n = 0, \pm 1, \pm 2, \ldots$.
3. The distance between any two successive maxima or minima is one-half wavelength, since

$$\beta z = n\pi \qquad z = \frac{n\pi}{\beta} = \frac{n\pi}{2\pi/\lambda} = n\frac{\lambda}{2} \qquad (n = 0, \pm 1, \pm 2, \ldots)$$

Then

$$z_1 = \frac{\lambda}{2} \tag{3-3-7}$$

It is evident that there are no zeros in the minimum. Similarly,

$$\mathbf{I}_{\text{max}} = \mathbf{I}_+ e^{-\alpha z} + \mathbf{I}_- e^{\alpha z} = \mathbf{I}_+ e^{-\alpha z}(1 + |\Gamma|) \tag{3-3-8}$$

$$\mathbf{I}_{\text{min}} = \mathbf{I}_+ e^{-\alpha z} - \mathbf{I}_- e^{\alpha z} = \mathbf{I}_+ e^{-\alpha z}(1 - |\Gamma|) \tag{3-3-9}$$

The standing-wave patterns of two oppositely traveling waves with unequal amplitude in lossy or lossless line are shown in Figs. 3-3-1 and 3-3-2.

A further study of Eq. (3-3-3) reveals that

1. When $\mathbf{V}_+ \neq 0$ and $\mathbf{V}_- = 0$, the standing-wave pattern becomes

$$\mathbf{V}_0 = \mathbf{V}_+ e^{-\alpha z} \tag{3-3-10}$$

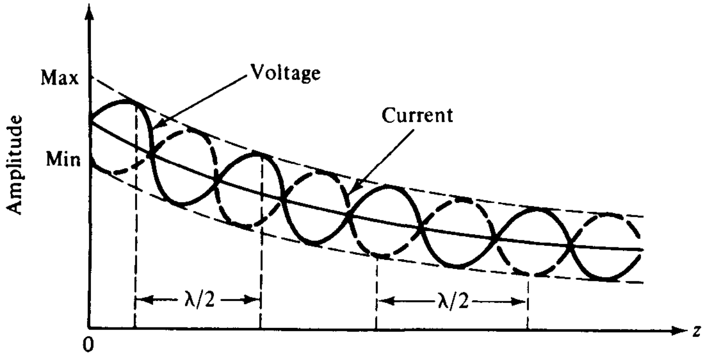


Figure 3-3-1 Standing-wave pattern in a lossy line.

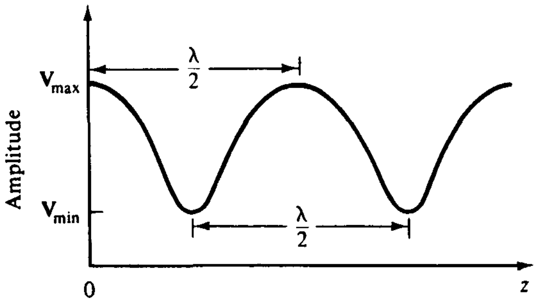


Figure 3-3-2 Voltage standing-wave pattern in a lossless line.

2. When $\mathbf{V}_+ = 0$ and $\mathbf{V}_- \neq 0$, the standing-wave pattern becomes

$$\mathbf{V}_0 = \mathbf{V}_- e^{\alpha z} \tag{3-3-11}$$

3. When the positive wave and the negative wave have equal amplitudes (that is, $|\mathbf{V}_+ e^{-\alpha z}| = |\mathbf{V}_- e^{\alpha z}|$) or the magnitude of the reflection coefficient is unity, the standing-wave pattern with a zero phase is given by

$$\mathbf{V}_s = 2\mathbf{V}_+ e^{-\alpha z} \cos(\beta z) \tag{3-3-12}$$

which is called a *pure standing wave*.

Similarly, the equation of a pure standing wave for the current is

$$\mathbf{I}_s = -j2\mathbf{Y}_0 \mathbf{V}_+ e^{-\alpha z} \sin(\beta z) \tag{3-3-13}$$

Equations (3-3-12) and (3-3-13) show that the voltage and current standing waves are 90° out of phase along the line. The points of zero current are called the *current nodes*. The voltage nodes and current nodes are interlaced a quarter wavelength apart.

The voltage and current may be expressed as real functions of time and space:

$$v_s = (z, t) = \mathrm{Re}[\mathbf{V}_s(z)e^{j\omega t}] = 2\mathbf{V}_+ e^{-\alpha z} \cos(\beta z) \cos(\omega t) \tag{3-3-14}$$

$$i_s = (z, t) = \mathrm{Re}[\mathbf{I}_s(z)e^{j\omega t}] = 2\mathbf{Y}_0 \mathbf{V}_+ e^{-\alpha z} \sin(\beta z) \sin(\omega t) \tag{3-3-15}$$

The amplitudes of Eqs. (3-3-14) and (3-3-15) vary sinusoidally with time; the

voltage is a maximum at the instant when the current is zero and vice versa. Figure 3-3-3 shows the pure-standing-wave patterns of the phasor of Eqs. (3-3-12) and (3-3-13) for an open-terminal line.

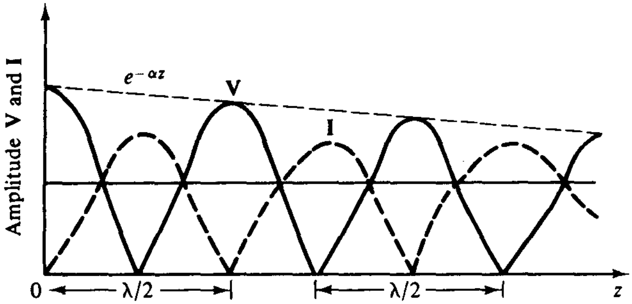


Figure 3-3-3 Pure standing waves of voltage and current.

3-3-2 Standing-Wave Ratio

Standing waves result from the simultaneous presence of waves traveling in opposite directions on a transmission line. The ratio of the maximum of the standing-wave pattern to the minimum is defined as the standing-wave ratio, designated by ρ. That is,

$$\text{Standing-wave ratio} \equiv \frac{\text{maximum voltage or current}}{\text{minimum voltage or current}}$$

$$\rho \equiv \frac{|\mathbf{V}_{\text{max}}|}{|\mathbf{V}_{\text{min}}|} = \frac{|\mathbf{I}_{\text{max}}|}{|\mathbf{I}_{\text{min}}|} \tag{3-3-16}$$

The standing-wave ratio results from the fact that the two traveling-wave components of Eq. (3-3-1) add in phase at some points and subtract at other points. The distance between two successive maxima or minima is $\lambda/2$. The standing-wave ratio of a pure traveling wave is unity and that of a pure standing wave is infinite. It should be noted that since the standing-wave ratios of voltage and current are identical, no distinctions are made between VSWR and ISWR.

When the standing-wave ratio is unity, there is no reflected wave and the line is called a *flat line*. The standing-wave ratio cannot be defined on a lossy line because the standing-wave pattern changes markedly from one position to another. On a low-loss line the ratio remains fairly constant, and it may be defined for some region. For a lossless line, the ratio stays the same throughout the line.

Since the reflected wave is defined as the product of an incident wave and its reflection coefficient, the standing-wave ratio ρ is related to the reflection coefficient $\mathbf{\Gamma}$ by

$$\rho = \frac{1 + |\mathbf{\Gamma}|}{1 - |\mathbf{\Gamma}|} \tag{3-3-17}$$

and vice versa

$$|\Gamma| = \frac{\rho - 1}{\rho + 1} \tag{3-3-18}$$

This relation is very useful for determining the reflection coefficient from the standing-wave ratio, which is usually found from the Smith chart. The curve in Fig. 3-3-4 shows the relationship between reflection coefficient $|\Gamma|$ and standing-wave ratio ρ.

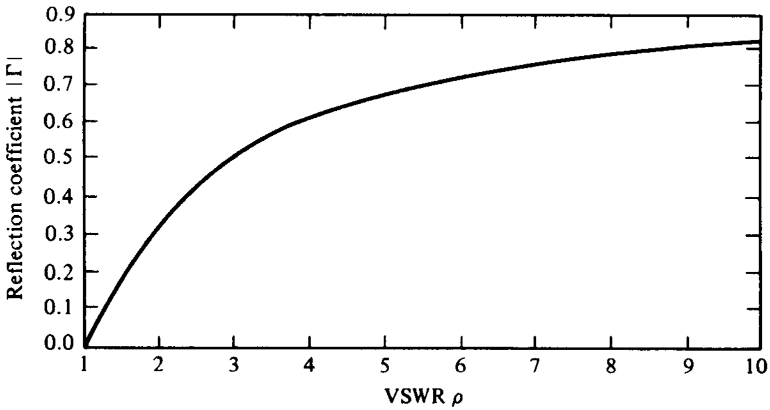


Figure 3-3-4 SWR versus reflection coefficient.

As a result of Eq. (3-3-17), since $|\Gamma| \leq 1$, the standing-wave ratio is a positive real number and never less than unity, $\rho \geq 1$. From Eq. (3-3-18) the magnitude of the reflection coefficient is never greater than unity.

Example 3-3-1: Standing-Wave Ratio

A transmission line has a characteristic impedance of $50 + j0.01\ \Omega$ and is terminated in a load impedance of $73 - j42.5\ \Omega$. Calculate: **(a)** the reflection coefficient; **(b)** the standing-wave ratio.

Solution

a. From Eq. (3-2-8) the reflection coefficient is

$$\Gamma = \frac{\mathbf{Z}_\ell - \mathbf{Z}_0}{\mathbf{Z}_\ell + \mathbf{Z}_0} = \frac{73 - j42.5 - (50 + j0.01)}{73 - j42.5 + (50 + j0.01)} = 0.377\underline{/-42.7^\circ}$$

b. From Eq. (3-3-17) the standing-wave ratio is

$$\rho = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.377}{1 - 0.377} = 2.21$$

3-4 LINE IMPEDANCE AND ADMITTANCE

3-4-1 Line Impedance

The line impedance of a transmission line is the complex ratio of the voltage phasor at any point to the current phasor at that point. It is defined as

$$\mathbf{Z} \equiv \frac{\mathbf{V}(z)}{\mathbf{I}(z)} \tag{3-4-1}$$

Figure 3-4-1 shows a diagram for a transmission line.

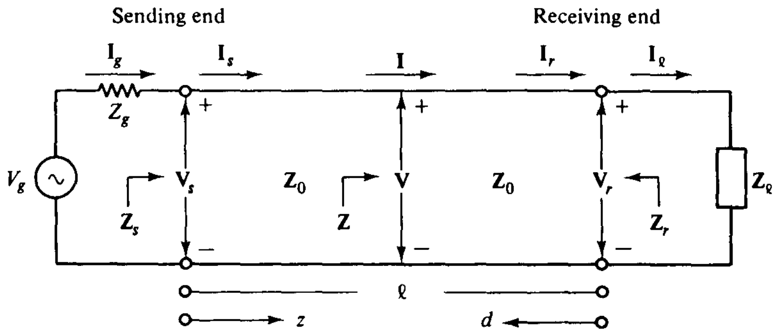


Figure 3-4-1 Diagram of a transmission line showing notations.

In general, the voltage or current along a line is the sum of the respective incident wave and reflected wave—that is,

$$\mathbf{V} = \mathbf{V}_{\text{inc}} + \mathbf{V}_{\text{ref}} = \mathbf{V}_+ e^{-\gamma z} + \mathbf{V}_- e^{\gamma z} \tag{3-4-2}$$

$$\mathbf{I} = \mathbf{I}_{\text{inc}} + \mathbf{I}_{\text{ref}} = \mathbf{Y}_0(\mathbf{V}_+ e^{-\gamma z} - \mathbf{V}_- e^{\gamma z}) \tag{3-4-3}$$

At the sending end $z = 0$; then Eqs. (3-4-2) and (3-4-3) become

$$\mathbf{I}_s \mathbf{Z}_s = \mathbf{V}_+ + \mathbf{V}_- \tag{3-4-4}$$

$$\mathbf{I}_s \mathbf{Z}_0 = \mathbf{V}_+ - \mathbf{V}_- \tag{3-4-5}$$

By solving these two equations for $\mathbf{V}_+$ and $\mathbf{V}_-$, we obtain

$$\mathbf{V}_+ = \frac{\mathbf{I}_s}{2}(\mathbf{Z}_s + \mathbf{Z}_0) \tag{3-4-6}$$

$$\mathbf{V}_- = \frac{\mathbf{I}_s}{2}(\mathbf{Z}_s - \mathbf{Z}_0) \tag{3-4-7}$$

Substitution of $\mathbf{V}_+$ and $\mathbf{V}_-$ in Eqs. (3-4-2) and (3-4-3) yields

$$\mathbf{V} = \frac{\mathbf{I}_s}{2}[(\mathbf{Z}_s + \mathbf{Z}_0)e^{-\gamma z} + (\mathbf{Z}_s - \mathbf{Z}_0)e^{\gamma z}] \tag{3-4-8}$$

$$\mathbf{I} = \frac{\mathbf{I}_s}{2\mathbf{Z}_0}[(\mathbf{Z}_s + \mathbf{Z}_0)e^{-\gamma z} - (\mathbf{Z}_s - \mathbf{Z}_0)e^{\gamma z}] \tag{3-4-9}$$

Then the line impedance at any point z from the sending end in terms of $\mathbf{Z}_s$ and $\mathbf{Z}_0$ is expressed as

$$\mathbf{Z} = \mathbf{Z}_0 \frac{(\mathbf{Z}_s + \mathbf{Z}_0)e^{-\gamma z} + (\mathbf{Z}_s - \mathbf{Z}_0)e^{\gamma z}}{(\mathbf{Z}_s + \mathbf{Z}_0)e^{-\gamma z} - (\mathbf{Z}_s - \mathbf{Z}_0)e^{\gamma z}} \tag{3-4-10}$$

At $z = \ell$ the line impedance at the receiving end in terms of $\mathbf{Z}_s$ and $\mathbf{Z}_0$ is given by

$$\mathbf{Z}_r = \mathbf{Z}_0 \frac{(\mathbf{Z}_s + \mathbf{Z}_0)e^{-\gamma \ell} + (\mathbf{Z}_s - \mathbf{Z}_0)e^{\gamma \ell}}{(\mathbf{Z}_s + \mathbf{Z}_0)e^{-\gamma \ell} - (\mathbf{Z}_s - \mathbf{Z}_0)e^{\gamma \ell}} \tag{3-4-11}$$

Alternatively, the line impedance can be expressed in terms of $\mathbf{Z}_\ell$ and $\mathbf{Z}_0$. At $z = \ell$, $\mathbf{V}_r = \mathbf{I}_\ell \mathbf{Z}_\ell$; then

$$\mathbf{I}_\ell \mathbf{Z}_\ell = \mathbf{V}_+ e^{-\gamma \ell} + \mathbf{V}_- e^{\gamma \ell} \tag{3-4-12}$$

$$\mathbf{I}_\ell \mathbf{Z}_0 = \mathbf{V}_+ e^{-\gamma \ell} - \mathbf{V}_- e^{\gamma \ell} \tag{3-4-13}$$

Solving these two equations for $\mathbf{V}_+$ and $\mathbf{V}_-$, we have

$$\mathbf{V}_+ = \frac{\mathbf{I}_\ell}{2}(\mathbf{Z}_\ell + \mathbf{Z}_0)e^{\gamma \ell} \tag{3-4-14}$$

$$\mathbf{V}_- = \frac{\mathbf{I}_\ell}{2}(\mathbf{Z}_\ell - \mathbf{Z}_0)e^{-\gamma \ell} \tag{3-4-15}$$

Then substituting these results in Eqs. (3-4-2) and (3-4-3) and letting $z = \ell - d$, we obtain

$$\mathbf{V} = \frac{\mathbf{I}_\ell}{2}[(\mathbf{Z}_\ell + \mathbf{Z}_0)e^{\gamma d} + (\mathbf{Z}_\ell - \mathbf{Z}_0)e^{-\gamma d}] \tag{3-4-16}$$

$$\mathbf{I} = \frac{\mathbf{I}_\ell}{2\mathbf{Z}_0}[(\mathbf{Z}_\ell + \mathbf{Z}_0)e^{\gamma d} - (\mathbf{Z}_\ell - \mathbf{Z}_0)e^{-\gamma d}] \tag{3-4-17}$$

Next, the line impedance at any point from the receiving end in terms of $\mathbf{Z}_\ell$ and $\mathbf{Z}_0$ is expressed as

$$\mathbf{Z} = \mathbf{Z}_0 \frac{(\mathbf{Z}_\ell + \mathbf{Z}_0)e^{\gamma d} + (\mathbf{Z}_\ell - \mathbf{Z}_0)e^{-\gamma d}}{(\mathbf{Z}_\ell + \mathbf{Z}_0)e^{\gamma d} - (\mathbf{Z}_\ell - \mathbf{Z}_0)e^{-\gamma d}} \tag{3-4-18}$$

The line impedance at the sending end can also be found from Eq. (3-4-18) by letting $d = \ell$:

$$\mathbf{Z}_s = \mathbf{Z}_0 \frac{(\mathbf{Z}_\ell + \mathbf{Z}_0)e^{\gamma \ell} + (\mathbf{Z}_\ell - \mathbf{Z}_0)e^{-\gamma \ell}}{(\mathbf{Z}_\ell + \mathbf{Z}_0)e^{\gamma \ell} - (\mathbf{Z}_\ell - \mathbf{Z}_0)e^{-\gamma \ell}} \tag{3-4-19}$$

It is a tedious task to solve Eqs. (3-4-10), (3-4-11), (3-4-18), or (3-4-19) for the line impedance. These equations can be simplified by replacing the exponential factors with either hyperbolic functions or circular functions. The hyperbolic functions are obtained from

$$e^{\pm \gamma z} = \cosh(\gamma z) \pm \sinh(\gamma z) \tag{3-4-20}$$

Substitution of the hyperbolic functions in Eq. (3-4-10) yields the line impedance at

any point from the sending end in terms of the hyperbolic functions:

$$\mathbf{Z} = \mathbf{Z}_0 \frac{\mathbf{Z}_s \cosh(\gamma z) - \mathbf{Z}_0 \sinh(\gamma z)}{\mathbf{Z}_0 \cosh(\gamma z) - \mathbf{Z}_s \sinh(\gamma z)} = \mathbf{Z}_0 \frac{\mathbf{Z}_s - \mathbf{Z}_0 \tanh(\gamma z)}{\mathbf{Z}_0 - \mathbf{Z}_s \tanh(\gamma z)} \tag{3-4-21}$$

Similarly, substitution of the hyperbolic functions in Eq. (3-4-18) yields the line impedance from the receiving end in terms of the hyperbolic function:

$$\mathbf{Z} = \mathbf{Z}_0 \frac{\mathbf{Z}_\ell \cosh(\gamma d) + \mathbf{Z}_0 \sinh(\gamma d)}{\mathbf{Z}_0 \cosh(\gamma d) + \mathbf{Z}_\ell \sinh(\gamma d)} = \mathbf{Z}_0 \frac{\mathbf{Z}_\ell + \mathbf{Z}_0 \tanh(\gamma d)}{\mathbf{Z}_0 + \mathbf{Z}_\ell \tanh(\gamma d)} \tag{3-4-22}$$

For a lossless line, $\gamma = j\beta$; and by using the following relationships between hyperbolic and circular functions

$$\sinh(j\beta z) = j \sin(\beta z) \tag{3-4-23}$$

$$\cosh(j\beta z) = \cos(\beta z) \tag{3-4-24}$$

the impedance of a lossless transmission line ($\mathbf{Z}_0 = R_0$) can be expressed in terms of the circular functions:

$$\mathbf{Z} = R_0 \frac{\mathbf{Z}_s \cos(\beta z) - jR_0 \sin(\beta z)}{R_0 \cos(\beta z) - j\mathbf{Z}_s \sin(\beta z)} = R_0 \frac{\mathbf{Z}_s - jR_0 \tan(\beta z)}{R_0 - j\mathbf{Z}_s \tan(\beta z)} \tag{3-4-25}$$

and

$$\mathbf{Z} = R_0 \frac{\mathbf{Z}_\ell \cos(\beta d) + jR_0 \sin(\beta d)}{R_0 \cos(\beta d) + j\mathbf{Z}_\ell \sin(\beta d)} = R_0 \frac{\mathbf{Z}_\ell + jR_0 \tan(\beta d)}{R_0 + j\mathbf{Z}_\ell \tan(\beta d)} \tag{3-4-26}$$

Impedance in terms of reflection coefficient or standing-wave ratio. Rearrangement of Eq. (3-4-18) gives the line impedance—looking at it from the receiving end—as

$$\mathbf{Z} = \mathbf{Z}_0 \frac{1 + \mathbf{\Gamma}_\ell e^{-2\gamma d}}{1 - \mathbf{\Gamma}_\ell e^{-2\gamma d}} \tag{3-4-27}$$

in which the following substitution is made by

$$\mathbf{\Gamma}_\ell = \frac{\mathbf{Z}_\ell - \mathbf{Z}_0}{\mathbf{Z}_\ell + \mathbf{Z}_0} \tag{3-4-28}$$

From Eq. (3-2-12) the reflection coefficient at a distance d from the receiving end is given by

$$\mathbf{\Gamma} = \mathbf{\Gamma}_\ell e^{-2\gamma d} = |\mathbf{\Gamma}_\ell| e^{-2\alpha d} e^{j(\theta_\ell - 2\beta d)} \tag{3-4-29}$$

Then the simple equation for the line impedance at a distance d from the load is expressed by

$$\mathbf{Z} = \mathbf{Z}_0 \frac{1 + \mathbf{\Gamma}}{1 - \mathbf{\Gamma}} \tag{3-4-30}$$

The reflected coefficient is usually a complex quantity and can be written

$$\mathbf{\Gamma} = |\mathbf{\Gamma}| e^{j\phi} \tag{3-4-31}$$

where $|\mathbf{\Gamma}| = |\mathbf{\Gamma}_\ell| e^{-2\alpha d}$
$\phi = \theta_\ell - 2\beta d$

The impedance variation along a lossless line can be found as follows:

$$\mathbf{Z}(d) = \mathbf{Z}_0 \frac{1 + |\mathbf{\Gamma}| e^{j\phi}}{1 - |\mathbf{\Gamma}| e^{j\phi}} = R_0 \frac{1 + |\mathbf{\Gamma}|(\cos\phi + j\sin\phi)}{1 - |\mathbf{\Gamma}|(\cos\phi + j\sin\phi)}$$

$$= R(d) + jX(d) = |\mathbf{Z}(d)| e^{j\theta_d} \tag{3-4-32)}$$

where
$$|\mathbf{Z}(d)| = R_0 \sqrt{\frac{1 + 2|\mathbf{\Gamma}|\cos\phi + |\mathbf{\Gamma}|^2}{1 - 2|\mathbf{\Gamma}|\cos\phi + |\mathbf{\Gamma}|^2}} \tag{3-4-33}$$

$$R(d) = R_0 \frac{1 - |\mathbf{\Gamma}|^2}{1 - 2|\mathbf{\Gamma}|\cos\phi + |\mathbf{\Gamma}|^2} \tag{3-4-34}$$

$$X(d) = R_0 \frac{2|\mathbf{\Gamma}|\sin\phi}{1 - 2|\mathbf{\Gamma}|\cos\phi + |\mathbf{\Gamma}|^2} \tag{3-4-35}$$

$$\theta_d = \arctan\left(\frac{X}{R}\right) = \arctan\left(\frac{2|\mathbf{\Gamma}|\sin\phi}{1 - |\mathbf{\Gamma}|^2}\right) \tag{3-4-36}$$

Since $\phi = \theta_\ell - 2\beta d$, then $\phi = \theta_\ell - 2\pi$ if $\beta d = \pi$. However, $\cos(\theta_\ell - 2\pi) = \cos\theta_\ell$ and $\sin(\theta_\ell - 2\pi) = \sin\theta_\ell$; then

$$\mathbf{Z}(d) = \mathbf{Z}\left(d + \frac{\pi}{\beta}\right) = \mathbf{Z}\left(d + \frac{\lambda}{2}\right) \tag{3-4-37}$$

It is concluded that the impedance along a lossless line will be repeated for every interval at a half-wavelength distance.

Furthermore, the magnitude of a reflection coefficient $|\mathbf{\Gamma}|$ is related to the standing-wave ratio ρ by

$$|\mathbf{\Gamma}| = \frac{\rho - 1}{\rho + 1} \quad \text{and} \quad \rho = \frac{1 + |\mathbf{\Gamma}|}{1 - |\mathbf{\Gamma}|} \tag{3-4-38}$$

The line impedance at any location from the receiving end can be written

$$\mathbf{Z} = R_0 \frac{(\rho + 1) + (\rho - 1)e^{j\phi}}{(\rho + 1) - (\rho - 1)e^{j\phi}} \tag{3-4-39}$$

This is a very useful equation for determining the line impedance in terms of standing-wave ratio ρ, since ρ can easily be measured by a detector or a standing-wave meter.

Determination of characteristic impedance. A common procedure for determining the characteristic impedance and propagation constant of a given transmission line is to take two measurements:

1. Measure the sending-end impedance with the receiving end short-circuited and record the result:

$$\mathbf{Z}_{sc} = \mathbf{Z}_0 \tanh(\gamma\ell) \tag{3-4-40}$$

2. Measure the sending-end impedance with the receiving end open-circuited and record the result:

$$\mathbf{Z}_{oc} = \mathbf{Z}_0 \coth(\gamma \ell) \tag{3-4-41}$$

Then the characteristic impedance of the measured transmission line is given by

$$\mathbf{Z}_0 = \sqrt{\mathbf{Z}_{sc}\mathbf{Z}_{oc}} \tag{3-4-42}$$

and the propagation constant of the line can be computed from

$$\gamma = \alpha + j\beta = \frac{1}{\ell} \operatorname{arctanh} \sqrt{\frac{\mathbf{Z}_{sc}}{\mathbf{Z}_{oc}}} \tag{3-4-43}$$

Normalized impedance. The normalized impedance of a transmission line is defined as

$$\mathbf{z} \equiv \frac{\mathbf{Z}}{\mathbf{Z}_0} = \frac{1 + \mathbf{\Gamma}}{1 - \mathbf{\Gamma}} = r \pm jx \tag{3-4-44}$$

It should be noted that the lowercase letters are commonly designated for normalized quantities in describing the distributed transmission-line circuits.

An examination of Eqs. (3-4-39), (3-4-40), and (3-4-44) shows that the normalized impedance for a lossless line has the following significant features:

1. The maximum normalized impedance is

$$z_{max} = \frac{Z_{max}}{R_0} = \frac{|V_{max}|}{R_0|I_{min}|} = \frac{1 + |\mathbf{\Gamma}|}{1 - |\mathbf{\Gamma}|} = \rho \tag{3-4-45}$$

Here z_{max} is a positive real value and it is equal to the standing-wave ratio ρ at the location of any maximum voltage on the line.

2. The minimum normalized impedance is

$$z_{min} = \frac{Z_{min}}{R_0} = \frac{|V_{min}|}{R_0|I_{max}|} = \frac{1 - |\mathbf{\Gamma}|}{1 + |\mathbf{\Gamma}|} = \frac{1}{\rho} \tag{3-4-46}$$

Here z_{min} is a positive real number also but equals the reciprocal of the standing-wave ratio at the location of any minimum voltage on the line.

3. For every interval of a half-wavelength distance along the line, z_{max} or z_{min} is repeated:

$$z_{max}(z) = z_{max}\left(z \pm \frac{\lambda}{2}\right) \tag{3-4-47}$$

$$z_{min}(z) = z_{min}\left(z \pm \frac{\lambda}{2}\right) \tag{3-4-48}$$

4. Since V_{max} and V_{min} are separated by a quarter-wavelength, z_{max} is equal to the reciprocal of z_{min} for every $\lambda/4$ separation:

$$z_{max}\left(z \pm \frac{\lambda}{4}\right) = \frac{1}{z_{min}(z)} \tag{3-4-49}$$

3-4-2 Line Admittance

When a transmission line is branched, it is better to solve the line equations for the line voltage, current, and transmitted power in terms of admittance rather than impedance. The characteristic admittance and the generalized admittance are defined as

$$\mathbf{Y}_0 = \frac{1}{\mathbf{Z}_0} = G_0 \pm jB_0 \tag{3-4-50}$$

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = G \pm jB \tag{3-4-51}$$

Then the normalized admittance can be written

$$\mathbf{y} \equiv \frac{\mathbf{Y}}{\mathbf{Y}_0} = \frac{\mathbf{Z}_0}{\mathbf{Z}} = \frac{1}{\mathbf{z}} = g \pm jb \tag{3-4-52}$$

Example 3-4-1: Line Impedance

A lossless line has a characteristic impedance of 50 Ω and is terminated in a load resistance of 75 Ω. The line is energized by a generator which has an output impedance of 50 Ω and an open-circuit output voltage of 30 V (rms). The line is assumed to be 2.25 wavelengths long. Determine:

a. The input impedance
b. The magnitude of the instantaneous load voltage
c. The instantaneous power delivered to the load

Solution

a. From Eq. (3-4-26) the line that is 2.25 wavelengths long looks like a quarter-wave line. Then

$$\beta d = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2}$$

From Eq. (3-4-26) the input impedance is

$$\mathbf{Z}_{\text{in}} = \frac{R_0^2}{R_\ell} = \frac{(50)^5}{75} = 33.33\ \Omega$$

b. The reflection coefficient is

$$\mathbf{\Gamma}_\ell = \frac{R_\ell - R_0}{R_\ell + R_0} = \frac{75 - 50}{75 + 50} = 0.20$$

Then the instantaneous voltage at the load is

$$\mathbf{V}_\ell = \mathbf{V}_+ e^{-j\beta\ell}(1 + \mathbf{\Gamma}_\ell) = 30(1 + 0.20) = 36\ \text{V}$$

c. The instantaneous power delivered to the load is

$$\mathbf{P}_\ell = \frac{(36)^2}{75} = 17.28\ \text{W}$$

3-5 SMITH CHART

Many of the computations required to solve transmission-line problems involve the use of rather complicated equations. The solution of such problems is tedious and difficult because the accurate manipulation of numerous equations is necessary. To simplify their solution, we need a graphic method of arriving at a quick answer.

A number of impedance charts have been designed to facilitate the graphic solution of transmission-line problems. Basically all the charts are derived from the fundamental relationships expressed in the transmission equations. The most popular chart is that developed by Phillip H. Smith [1]. The purpose of this section is to present the graphic solutions of transmission-line problems by using the Smith chart.

The Smith chart consists of a plot of the normalized impedance or admittance with the angle and magnitude of a generalized complex reflection coefficient in a unity circle. The chart is applicable to the analysis of a lossless line as well as a lossy line. By simple rotation of the chart, the effect of the position on the line can be determined. To see how a Smith chart works, consider the equation of reflection coefficient at the load for a transmission line as shown in Eq. (3-2-8):

$$\mathbf{\Gamma}_\ell = \frac{\mathbf{Z}_\ell - \mathbf{Z}_0}{\mathbf{Z}_\ell + \mathbf{Z}_0} = |\mathbf{\Gamma}_\ell| e^{j\theta_\ell} = \Gamma_\gamma + j\Gamma_i \tag{3-5-1}$$

Since $|\Gamma_\ell| \leq 1$, the value of Γ_ℓ must lie on or within the unity circle with a radius of 1. The reflection coefficient at any other location along a line as shown in Eq. (3-2-12) is

$$\mathbf{\Gamma}_d = \mathbf{\Gamma}_\ell e^{-2\alpha d} e^{-j2\beta d} = |\mathbf{\Gamma}_\ell| e^{-2\alpha d} e^{j(\theta_\ell - 2\beta d)} \tag{3-5-2}$$

which is also on or within the unity circle. Figure 3-5-1 shows circles for a constant reflection coefficient $\mathbf{\Gamma}$ and constant electrical-length radials βd.

From Eqs. (3-4-29) and (3-4-44) the normalized impedance along a line is given by

$$\mathbf{z} = \frac{\mathbf{Z}}{\mathbf{Z}_0} = \frac{1 + \mathbf{\Gamma}_\ell e^{-2\gamma d}}{1 - \mathbf{\Gamma}_\ell e^{-2\gamma d}} \tag{3-5-3}$$

With no loss in generality, it is assumed that $d = 0$; then

$$\mathbf{z} = \frac{1 + \mathbf{\Gamma}_\ell}{1 - \mathbf{\Gamma}_\ell} = \frac{\mathbf{Z}_\ell}{\mathbf{Z}_0} = \frac{R_\ell + jX_\ell}{\mathbf{Z}_0} = r + jx \tag{3-5-4}$$

and

$$\mathbf{\Gamma}_\ell = \frac{\mathbf{z} - 1}{\mathbf{z} + 1} = \Gamma_r + j\Gamma_i \tag{3-5-5}$$

Substitution of Eq. (3-5-5) into Eq. (3-5-4) yields

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \tag{3-5-6}$$

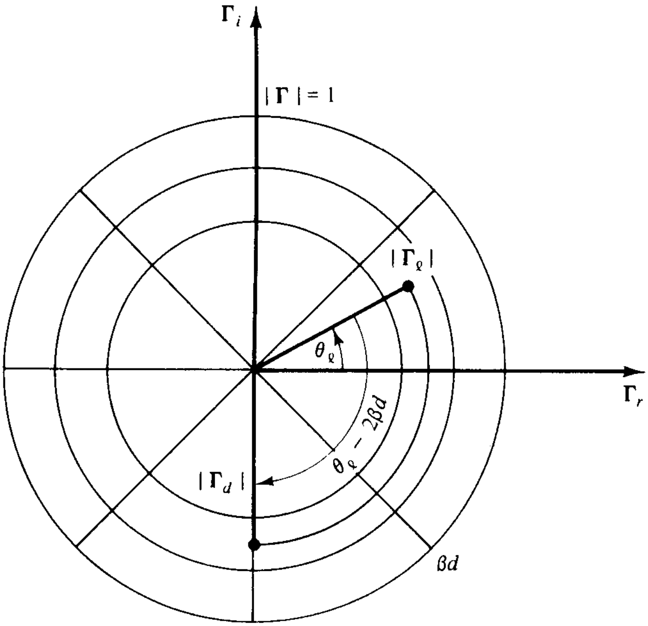


Figure 3-5-1 Constant Γ circles and electrical-length radials βd.

and

$$x = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \tag{3-5-7}$$

Equations (3-5-6) and (3-5-7) can be rearranged as

$$\left(\Gamma_r - \frac{r}{1 + r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1 + r}\right)^2 \tag{3-5-8}$$

and

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \tag{3-5-9}$$

Equation (3-5-8) represents a family of circles in which each circle has a constant resistance r. The radius of any circle is $1/(1 + r)$, and the center of any circle is $r/(1 + r)$ along the real axis in the unity circle, where r varies from zero to infinity. All constant resistance circles are plotted in Fig. 3-5-2 according to Eq. (3-5-8).

Equation (3-5-9) also describes a family of circles, but each of these circles specifies a constant reactance x. The radius of any circle is $(1/x)$, and the center of any circle is at

$$\Gamma_r = 1 \qquad \Gamma_i = \frac{1}{x} \qquad (\text{where } -\infty \leq x \leq \infty)$$

All constant reactance circles are plotted in Fig. 3-5-3 according to Eq. (3-5-9).

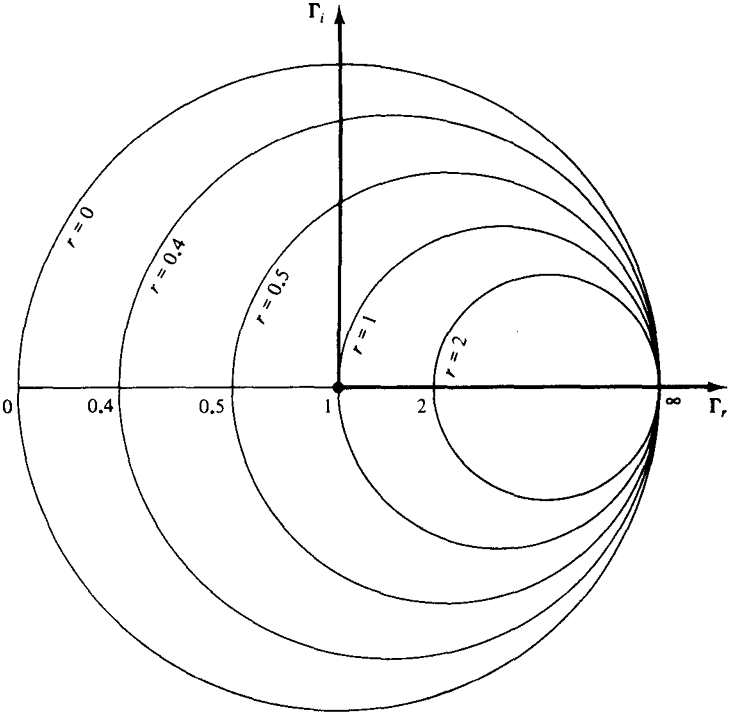


Figure 3-5-2 Constant resistance r circles.

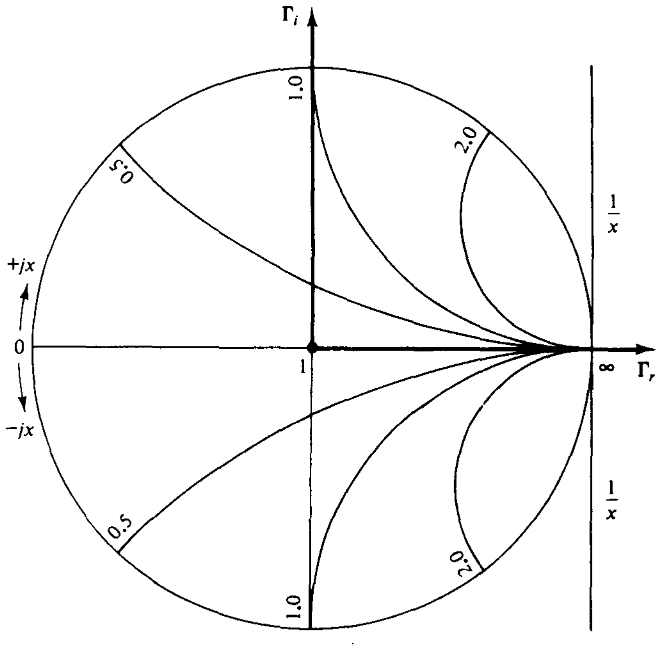


Figure 3-5-3 Constant reactance x circles.

There are relative distance scales in wavelength along the circumference of the Smith chart. Also, there is a phase scale specifying the angle of the reflection coefficient. When a normalized impedance **z** is located on the chart, the normalized impedance of any other location along the line can be found by use of Eq. (3-5-3):

$$\mathbf{z} = \frac{1 + \mathbf{\Gamma}_\ell e^{-2\gamma d}}{1 - \mathbf{\Gamma}_\ell e^{-2\gamma d}} \tag{3-5-10}$$

where

$$\mathbf{\Gamma}_\ell e^{-2\gamma d} = |\mathbf{\Gamma}_\ell| e^{-2\alpha d} e^{j(\theta_\ell - 2\beta d)} \tag{3-5-11}$$

The Smith chart may also be used for normalized admittance. This is evident since

$$\mathbf{Y}_0 = \frac{1}{\mathbf{Z}_0} = G_0 + jB_0 \quad \text{and} \quad \mathbf{Y} = \frac{1}{\mathbf{Z}} = G + jB \tag{3-5-12}$$

Then the normalized admittance is

$$\mathbf{y} = \frac{\mathbf{Y}}{\mathbf{Y}_0} = \frac{\mathbf{Z}_0}{\mathbf{Z}} = \frac{1}{\mathbf{z}} = g + jb \tag{3-5-13}$$

Figure 3-5-4 shows a Smith chart which superimposes Figs. 3-5-2 and 3-5-3 into one chart. The characteristics of the Smith chart are summarized as follows:

1. The constant r and constant x loci form two families of orthogonal circles in the chart.
2. The constant r and constant x circles all pass through the point ($\Gamma_r = 1$, $\Gamma_i = 0$).
3. The upper half of the diagram represents $+jx$.
4. The lower half of the diagram represents $-jx$.
5. For admittance the constant r circles become constant g circles, and the constant x circles become constant susceptance b circles.
6. The distance around the Smith chart once is one-half wavelength ($\lambda/2$).
7. At a point of $z_{\min} = 1/\rho$, there is a $V_{\min}$ on the line.
8. At a point of $z_{\max} = \rho$, there is a $V_{\max}$ on the line.
9. The horizontal radius to the right of the chart center corresponds to $V_{\max}$, $I_{\min}$, $z_{\max}$, and ρ (SWR).
10. The horizontal radius to the left of the chart center corresponds to $V_{\min}$, $I_{\max}$, $z_{\min}$, and $1/\rho$.
11. Since the normalized admittance **y** is a reciprocal of the normalized impedance **z**, the corresponding quantities in the admittance chart are 180° out of phase with those in the impedance chart.
12. The normalized impedance or admittance is repeated for every half wavelength of distance.
13. The distances are given in wavelengths toward the generator and also toward the load.

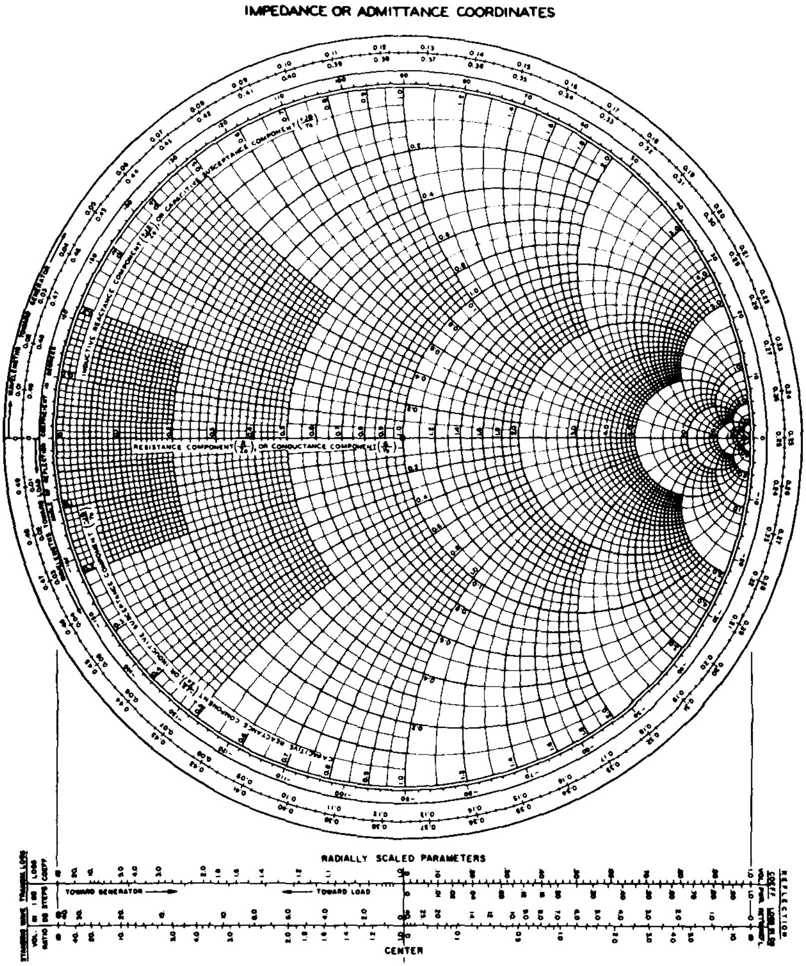


Figure 3-5-4 Smith chart.

The magnitude of the reflection coefficient is related to the standing-wave ratio by the following expression:

$$|\Gamma| = \frac{\rho - 1}{\rho + 1} \tag{3-5-14}$$

A Smith chart or slotted line can be used to measure a standing-wave pattern directly and then the magnitudes of the reflection coefficient, reflected power, transmitted power, and the load impedance can be calculated from it. The use of the Smith chart is illustrated in the following examples.

Example 3-5-1: Location Determination of Voltage Maximum and Minimum from Load

Given the normalized load impedance $z_\ell = 1 + j1$ and the operating wavelength

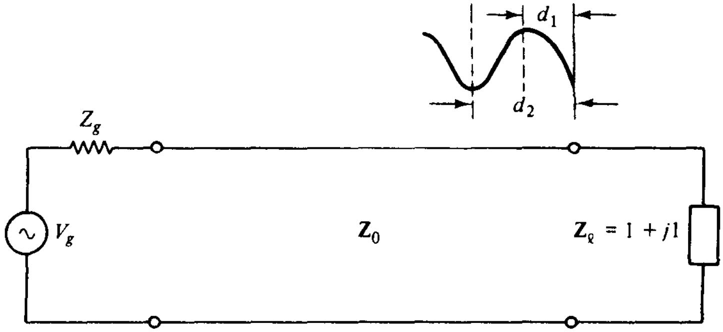


Figure 3-5-5 Diagram for Example 3-5-1.

$\lambda = 5$ cm, determine the first V_{max}, first V_{min} from the load, and the VSWR ρ as shown in Fig. 3-5-5.

Solution

1. Enter $z_\ell = 1 + j1$ on the chart as shown in Fig. 3-5-6.
2. Read 0.162λ on the distance scale by drawing a dashed-straight line from the center of the chart through the load point and intersecting the distance scale.
3. Move a distance from the point at 0.162λ toward the generator and first stop at the voltage maximum on the right-hand real axis at 0.25λ. Then

$$d_1(V_{max}) = (0.25 - 0.162)\lambda = (0.088)(5) = 0.44 \text{ cm}$$

4. Similarly, move a distance from the point of 0.162λ toward the generator and first stop at the voltage minimum on the left-hand real axis at 0.5λ. Then

$$d_2(V_{min}) = (0.5 - 0.162)\lambda = (0.338)(5) = 1.69 \text{ cm}$$

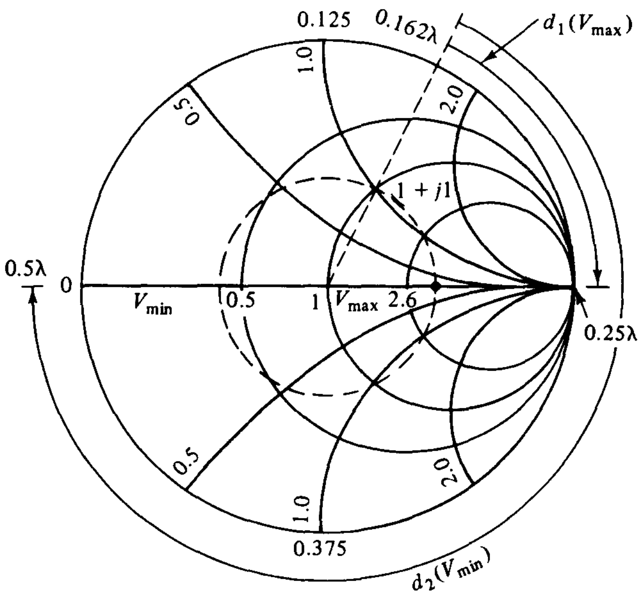


Figure 3-5-6 Graphic solution for Example 3-5-1.

5. Make a standing-wave circle with the center at (1, 0) and pass the circle through the point of $1 + j1$. The location intersected by the circle at the right portion of the real axis indicates the SWR. This is $\rho = 2.6$.

Example 3-5-2: Impedance Determination with Short-Circuit Minima Shift

The location of a minimum instead of a maximum is usually specified because it can be determined more accurately. Suppose that the characteristic impedance of the line R_0 is 50 Ω, and the SWR $\rho = 2$ when the line is loaded. When the load is shorted, the minima shift 0.15λ toward the load. Determine the load impedance. Figure 3-5-7 shows the diagram for the example.

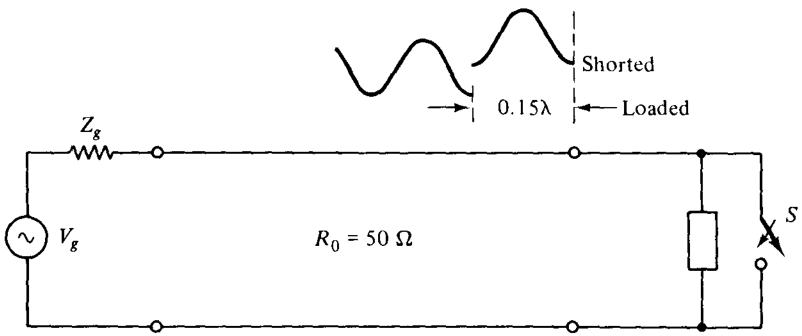


Figure 3-5-7 Diagram for Example 3-5-2.

Solution

1. When the line is shorted, the first voltage minimum occurs at the place of the load as shown in Fig. 3-5-8.

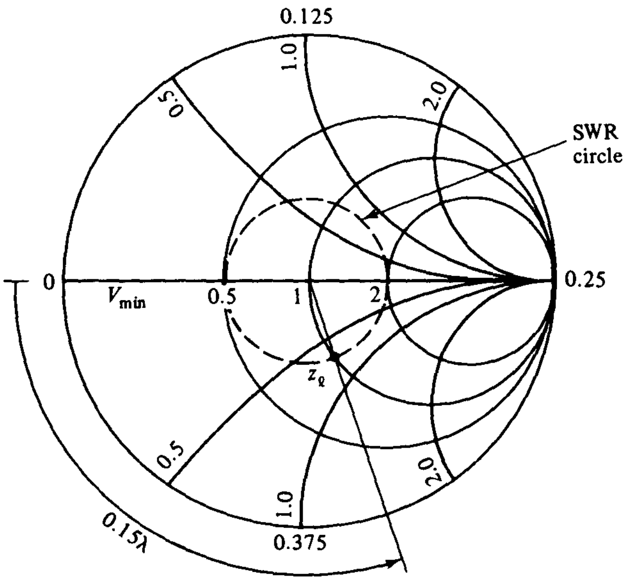


Figure 3-5-8 Graphic solution for Example 3-5-2.

2. When the line is loaded, the first voltage minimum shifts 0.15λ from the load. The distance between two successive minima is one-half wavelength.
3. Plot a SWR circle for $\rho = 2$.
4. Move a distance of 0.15λ from the minimum point along the distance scale toward the load and stop at 0.15λ.
5. Draw a line from this point to the center of the chart.
6. The intersection between the line and the SWR circle is

$$z_\ell = 1 - j0.65$$

7. The load impedance is

$$\mathbf{Z}_\ell = (1 - j0.65)(50) = 50 - j32.5\ \Omega$$

3-6 IMPEDANCE MATCHING

Impedance matching is very desirable with radio frequency (RF) transmission lines. Standing waves lead to increased losses and frequently cause the transmitter to malfunction. A line terminated in its characteristic impedance has a standing-wave ratio of unity and transmits a given power without reflection. Also, transmission efficiency is optimum where there is no reflected power. A "flat" line is nonresonant; that is, its input impedance always remains at the same value $\mathbf{Z}_0$ when the frequency changes.

Matching a transmission line has a special meaning, one differing from that used in circuit theory to indicate equal impedance seen looking both directions from a given terminal pair for maximum power transfer. In circuit theory, maximum power transfer requires the load impedance to be equal to the complex conjugate of the generator. This condition is sometimes referred to as a *conjugate match*. In transmission-line problems *matching* means simply terminating the line in its characteristic impedance.

A common application of RF transmission lines is the one in which there is a feeder connection between a transmitter and an antenna. Usually the input impedance to the antenna itself is not equal to the characteristic impedance of the line. Furthermore, the output impedance of the transmitter may not be equal to the $\mathbf{Z}_0$ of the line. Matching devices are necessary to flatten the line. A complete matched transmission-line system is shown in Fig. 3-6-1.

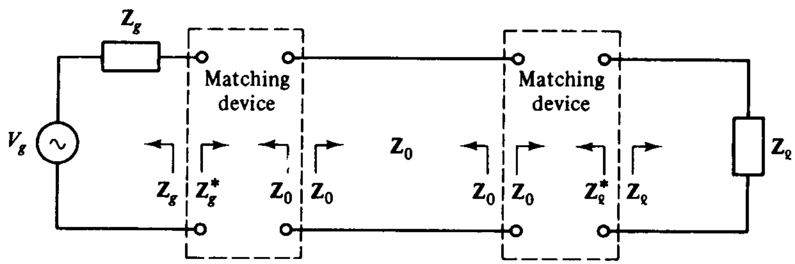


Figure 3-6-1 Matched transmission-line system.

For a low-loss or lossless transmission line at radio frequency, the characteristic impedance $\mathbf{Z}_0$ of the line is resistive. At every point the impedances looking in opposite directions are conjugate. If $\mathbf{Z}_0$ is real, it is its own conjugate. Matching can be tried first on the load side to flatten the line; then adjustment may be made on the transmitter side to provide maximum power transfer. At audio frequencies an iron-cored transformer is almost universally used as an impedance-matching device. Occasionally an iron-cored transformer is also used at radio frequencies. In a practical transmission-line system, the transmitter is ordinarily matched to the coaxial cable for maximum power transfer. Because of the variable loads, however, an impedance-matching technique is often required at the load side.

Since the matching problems involve parallel connections on the transmission line, it is necessary to work out the problems with admittances rather than impedances. The Smith chart itself can be used as a computer to convert the normalized impedance to admittance by a rotation of 180°, as described earlier.

3-6-1 Single-Stub Matching

Although single-lumped inductors or capacitors can match the transmission line, it is more common to use the susceptive properties of short-circuited sections of transmission lines. Short-circuited sections are preferable to open-circuited ones because a good short circuit is easier to obtain than a good open circuit.

For a lossless line with $\mathbf{Y}_g = \mathbf{Y}_0$, maximum power transfer requires $\mathbf{Y}_{11} = \mathbf{Y}_0$, where $\mathbf{Y}_{11}$ is the total admittance of the line and stub looking to the right at point 1-1 (see Fig. 3-6-2). The stub must be located at that point on the line where the real part of the admittance, looking toward the load, is $\mathbf{Y}_0$. In a normalized unit y_{11} must be in the form

$$y_{11} = y_d \pm y_s = 1 \tag{3-6-1}$$

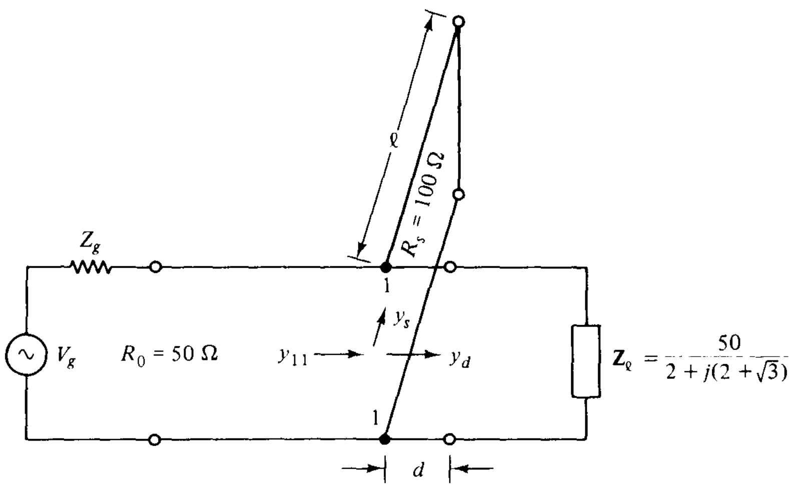


Figure 3-6-2 Single-stub matching for Example 3-6-1.

if the stub has the same characteristic impedance as that of the line. Otherwise

$$\mathbf{Y}_{11} = \mathbf{Y}_d \pm \mathbf{Y}_s = \mathbf{Y}_0 \tag{3-6-2}$$

The stub length is then adjusted so that its susceptance just cancels out the susceptance of the line at the junction.

Example 3-6-1: Single-Stub Matching

A lossless line of characteristic impedance $R_0 = 50\ \Omega$ is to be matched to a load $\mathbf{Z}_\ell = 50/[2 + j(2 + \sqrt{3})]\ \Omega$ by means of a lossless short-circuited stub. The characteristic impedance of the stub is $100\ \Omega$. Find the stub position (closest to the load) and length so that a match is obtained.

Solution

1. Compute the normalized load admittance and enter it on the Smith chart (see Fig. 3-6-3).

$$y_\ell = \frac{1}{z_\ell} = \frac{R_0}{\mathbf{Z}_\ell} = 2 + j(2 + \sqrt{3}) = 2 + j3.732$$

2. Draw a SWR circle through the point of y_ℓ so that the circle intersects the unity circle at the point y_d.

$$y_d = 1 - j2.6$$

Note that there are an infinite number of y_d. Take the one that allows the stub to be attached as closely as possible to the load.

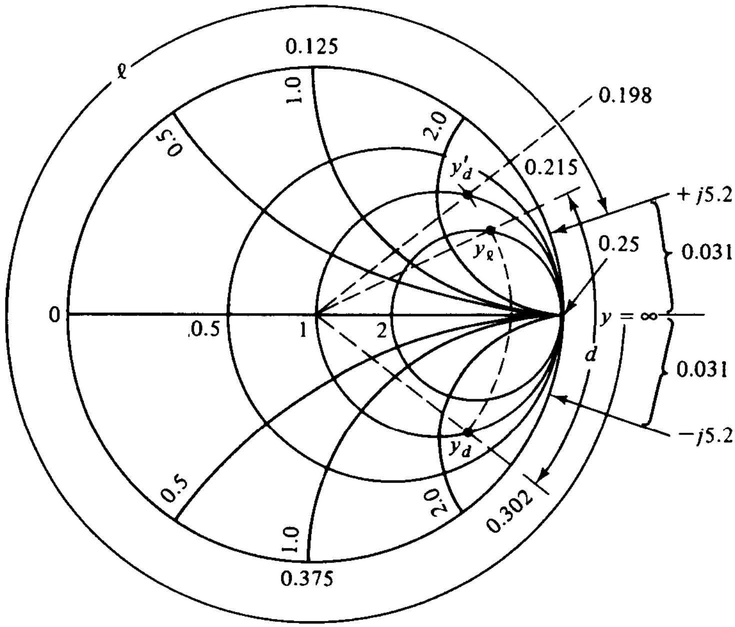


Figure 3-6-3 Graphic solution for Example 3-6-1.

3. Since the characteristic impedance of the stub is different from that of the line, the condition for impedance matching at the junction requires

$$\mathbf{Y}_{11} = \mathbf{Y}_d + \mathbf{Y}_s$$

where $\mathbf{Y}_s$ is the susceptance that the stub will contribute.

It is clear that the stub and the portion of the line from the load to the junction are in parallel, as seen by the main line extending to the generator. The admittances must be converted to normalized values for matching on the Smith chart. Then Eq. (3-6-2) becomes

$$y_{11}\mathbf{Y}_0 = y_d\mathbf{Y}_0 + y_s\mathbf{Y}_{0s}$$

$$y_s = (y_{11} - y_d)\left(\frac{\mathbf{Y}_0}{\mathbf{Y}_{0s}}\right) = [1 - (1 - j2.6)]\frac{100}{50} = +j5.20$$

4. The distance between the load and the stub position can be calculated from the distance scale as

$$d = (0.302 - 0.215)\lambda = 0.087\lambda$$

5. Since the stub contributes a susceptance of $+j5.20$, enter $+j5.20$ on the chart and determine the required distance ℓ from the short-circuited end ($z = 0$, $y = \infty$), which corresponds to the right side of the real axis on the chart, by transversing the chart toward the generator until the point of $+j5.20$ is reached. Then

$$\ell = (0.50 - 0.031)\lambda = 0.469\lambda$$

When a line is matched at the junction, there will be no standing wave in the line from the stub to the generator.

6. If an inductive stub is required,

$$y'_d = 1 + j2.6$$

the susceptance of the stub will be

$$y'_s = -j5.2$$

7. The position of the stub from the load is

$$d' = [0.50 - (0.215 - 0.198)]\lambda = 0.483\lambda$$

and the length of the short-circuited stub is

$$\ell' = 0.031\lambda$$

3-6-2 Double-Stub Matching

Since single-stub matching is sometimes impractical because the stub cannot be placed physically in the ideal location, double-stub matching is needed. Double-stub devices consist of two short-circuited stubs connected in parallel with a fixed length between them. The length of the fixed section is usually one-eighth, three-eighths, or five-eighths of a wavelength. The stub that is nearest the load is used to adjust the susceptance and is located at a fixed wavelength from the constant conductance

unity circle ($g = 1$) on an appropriate constant-standing-wave-ratio circle. Then the admittance of the line at the second stub as shown in Fig. 3-6-4 is

$$y_{22} = y_{d2} \pm y_{s2} = 1 \tag{3-6-3}$$

$$\mathbf{Y}_{22} = \mathbf{Y}_{d2} \pm \mathbf{Y}_{s2} = \mathbf{Y}_0 \tag{3-6-4}$$

In these two equations it is assumed that the stubs and the main line have the same characteristic admittance. If the positions and lengths of the stubs are chosen properly, there will be no standing wave on the line to the left of the second stub measured from the load.

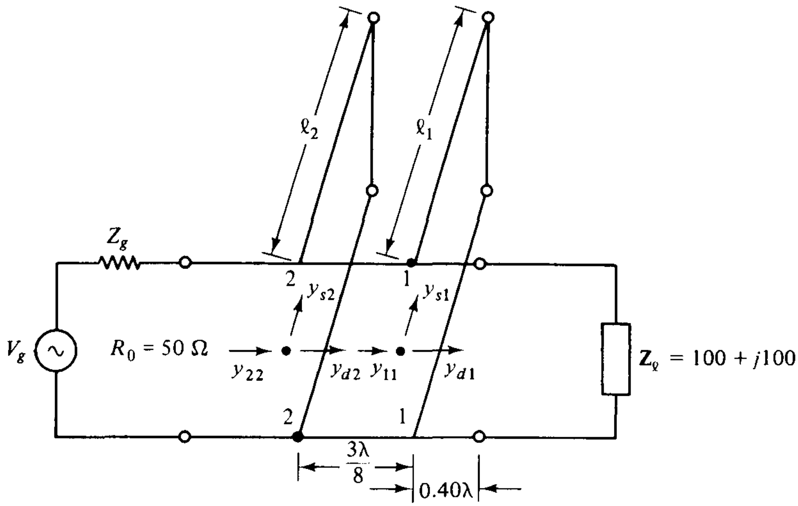


Figure 3-6-4 Double-stub matching for Example 3-6-2.

Example 3-6-2: Double-Stub Matching

The terminating impedance $\mathbf{Z}_\ell$ is $100 + j100\ \Omega$, and the characteristic impedance $\mathbf{Z}_0$ of the line and stub is $50\ \Omega$. The first stub is placed at 0.40λ away from the load. The spacing between the two stubs is $\frac{3}{8}\lambda$. Determine the length of the short-circuited stubs when the match is achieved. What terminations are forbidden for matching the line by the double-stub device?

Solution

1. Compute the normalized load impedance z_ℓ and enter it on the chart as shown in Fig. 3-6-5:

$$z_\ell = \frac{100 + j100}{50} = 2 + j2$$

2. Plot a SWR ρ circle and read the normalized load admittance 180° out of phase with z_ℓ on the SWR circle:

$$y_\ell = 0.25 - j0.25$$

3. Draw the spacing circle of $\frac{3}{8}\ell$ by rotating the constant-conductance unity circle

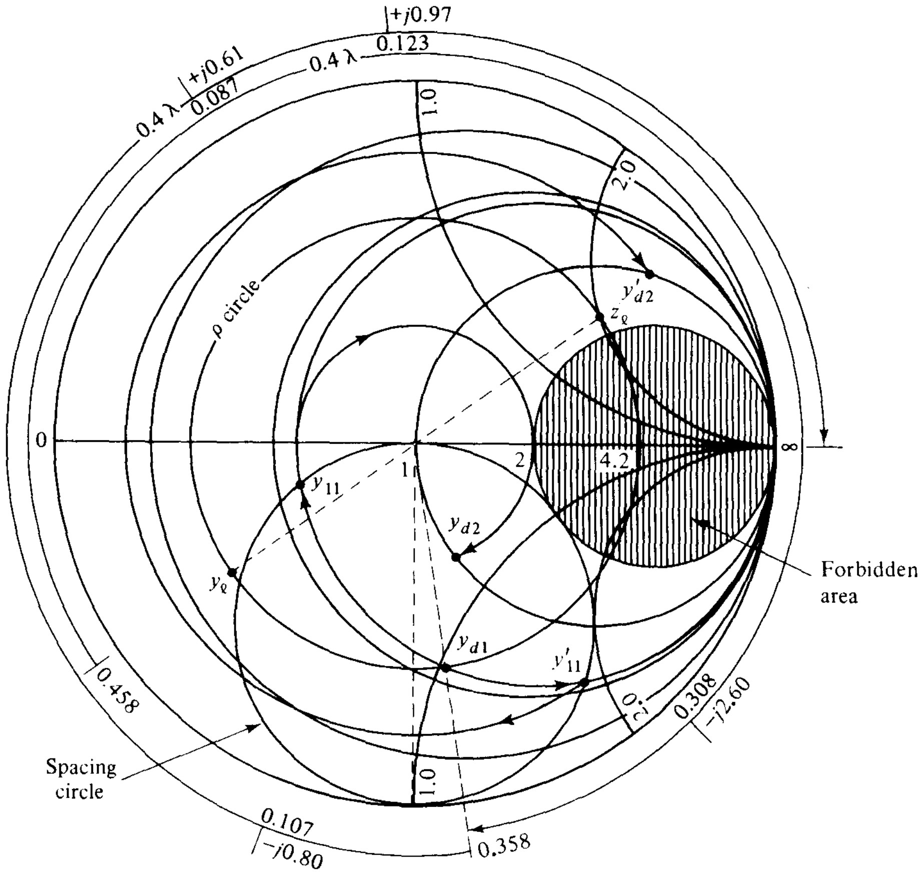


Figure 3-6-5 Graphic solution for Example 3-6-2.

(g = 1) through a phase angle of $2\beta d = 2\beta\frac{3}{8}\lambda = \frac{3}{2}\pi$ toward the load. Now y_{11} must be on this spacing circle, since y_{d2} will be on the $g = 1$ circle (y_{11} and y_{d2} are $\frac{3}{8}\lambda$ apart).

4. Move y_ℓ for a distance of 0.40λ from 0.458 to 0.358 along the SWR ρ circle toward the generator and read y_{d1} on the chart:

$$y_{d1} = 0.55 - j1.08$$

5. There are two possible solutions for y_{11}. They can be found by carrying y_{d1} along the constant-conductance ($g = 0.55$) circle that intersects the spacing circle at two points:

$$y_{11} = 0.55 - j0.11$$

$$y'_{11} = 0.55 - j1.88$$

6. At the junction 1-1,

$$y_{11} = y_{d1} + y_{s1}$$

Then

$$y_{s1} = y_{11} - y_{d1} = (0.55 - j0.11) - (0.55 - j1.08) = +j0.97$$

Similarly,

$$y'_{s1} = -j.080$$

7. The lengths of stub 1 are found as

$$\ell_1 = (0.25 + 0.123)\lambda = 0.373\lambda$$

$$\ell'_1 = (0.25 - 0.107)\lambda = 0.143\lambda$$

8. The $\frac{3}{8}\lambda$ section of line transforms y_{11} to y_{d2} and y_{11} to y'_{d2} along their constant standing-wave circles, respectively. That is,

$$y_{d2} = 1 - j0.61$$

$$y'_{d2} = 1 + j2.60$$

9. Then stub 2 must contribute

$$y_{s2} = +j0.61$$

$$y'_{s2} = -j2.60$$

10. The lengths of stub 2 are found as

$$\ell_2 = (0.25 + 0.087)\lambda = 0.337\lambda$$

$$\ell'_2 = (0.308 - 0.25)\lambda = 0.058\lambda$$

11. It can be seen from Fig. 3-6-5 that a normalized admittance y_ℓ located inside the hatched area cannot be brought to lie on the locus of y_{11} or y'_{11} for a possible match by the parallel connection of any short-circuited stub because the spacing circle and $g = 2$ circle are mutually tangent. Thus the area of a $g = 2$ circle is called the *forbidden region* of the normalized load admittance for possible match.

Normally the solution of a double-stub-matching problem can be worked out backward from the load toward the generator, since the load is known and the distance of the first stub away from the load can be arbitrarily chosen. In quite a few practical matching problems, however, some stubs have a different $\mathbf{Z}_0$ from that of the line, the length of a stub may be fixed, and so on. So it is hard to describe a definite procedure for solving the double-matching problems.

The flexible coaxial lines are available in different types. Their diameters vary from 0.635 cm (0.25 in.) to about 2.54 cm (1 in.), depending on the power requirement. In some coaxial cables, the inner conductor is stranded or a solid wire, but the outer conductor is a single braid or double. The dielectric material used in these coaxial lines is polyethylene, which has low loss at radio frequencies. Particularly for the RG series, the dielectric is either solid or foam polyethylene. The loss per unit length for foam polyethylene is even appreciably less than the equivalent solid polyethylene.

3-7 MICROWAVE COAXIAL CONNECTORS

For high-frequency operation the average circumference of a coaxial cable must be limited to about one wavelength, in order to reduce multimodal propagation and eliminate erratic reflection coefficients, power losses, and signal distortion. The standardization of coaxial connectors during World War II was mandatory for microwave operation to maintain a low reflection coefficient or a low voltage standing-wave ratio (VSWR). Since that time many modifications and new designs for microwave connectors have been proposed and developed. Seven types of microwave coaxial connectors are described below (see Fig. 3-7-1; TNC is not shown).

1. **APC–3.5** The APC–3.5 (Amphenol Precision Connector-3.5 mm) was originally developed by Hewlett-Packard, but is now manufactured by Amphenol. The connector provides repeatable connections and has a very low voltage standing-wave ratio (VSWR). Either the male or female end of this 50-ohm connector can mate with the opposite type of SMA connector. The APC–3.5 connector can work at frequencies up to 34 GHz.
2. **APC–7** The APC–7 (Amphenol Precision Connector-7 mm) was also developed by Hewlett-Packard in the mid 1960s, but it was recently improved and is now manufactured by Amphenol. The connector provides a coupling mechanism without male or female distinction and is the most repeatable connecting device used for very accurate 50-ohm measurement applications. Its VSWR is extremely low, in the range of 1.02 to 18 GHz. Maury Microwave also has an MPC series available.
3. **BNC** The BNC (Bayonet Navy Connector) was originally designed for military system applications during World War II. The connector operates very well at frequencies up to about 4 GHz; beyond that it tends to radiate electromagnetic energy. The BNC can accept flexible cables with diameters of up to 6.35 mm (0.25 in.) and characteristic impedance of 50 to 75 ohms. It is now the most commonly used connector for frequencies under 1 GHz.
4. **SMA** The SMA (Sub-Miniature A) connector was originally designed by Bendix Scintilla Corporation, but it has been manufactured by Omni-Spectra Inc. (as the OSM connector) and many other electronic companies. The main application of SMA connectors is on components for microwave systems. The connector is seldom used above 24 GHz because of higher-order modes.
5. **SMC** The SMC (Sub-Miniature C) is a 50-ohm connector that is smaller than the SMA. The connector is manufactured by Sealectro Corporation and can accept flexible cables with diameters of up to 3.17 mm (0.125 in.) for a frequency range of up to 7 GHz.
6. **TNC** The TNC (Threaded Navy Connector) is merely a threaded BNC. The function of the thread is to stop radiation at higher frequencies, so that the connector can work at frequencies up to 12 GHz.
7. **Type N** The Type N (Navy) connector was originally designed for military systems during World War II and is the most popular measurement connector

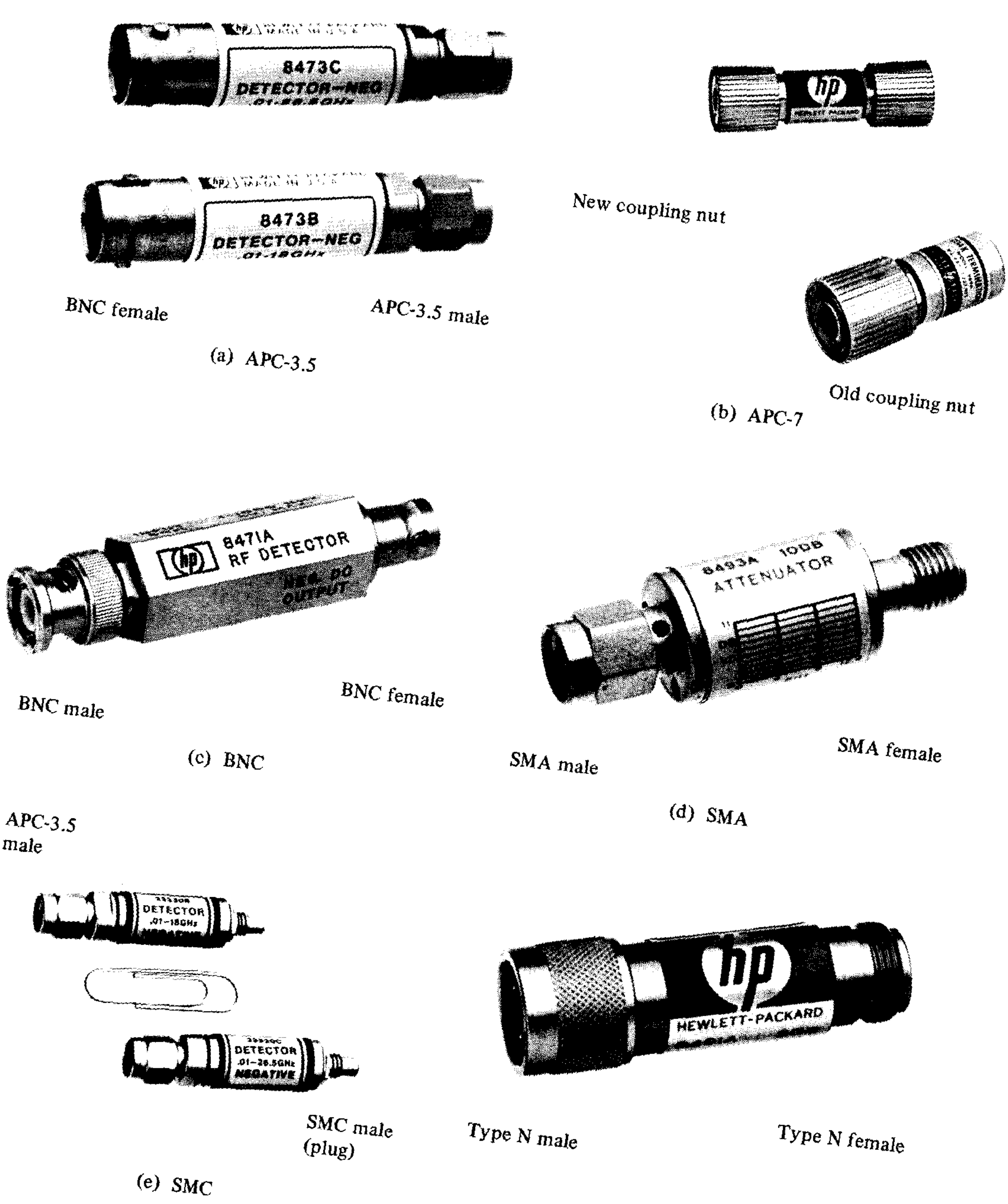


Figure 3-7-1 Microwave coaxial connectors.

for the frequency range of 1 to 18 GHz. It is 50- or 75-ohm connector and its VSWR is extremely low, less than 1.02.

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PROBLEMS

3-1. A transmission line has a characteristic impedance of 300 Ω and is terminated in a load of $300 - j300$ Ω. The propagation constant of the line is $0.054 + j3.53$ per meter. Determine:
- **a.** The reflection coefficient at the load
- **b.** The transition coefficient at the load
- **c.** The reflection coefficient at a point 2 m away from the load

3-2. A lossless transmission line has a characteristic impedance of 50 Ω and is terminated in a load of 100 Ω. The magnitude of a voltage wave incident to the line is 20 V (rms). Determine:
- **a.** The VSWR on the line
- **b.** The maximum voltage V_{max} and minimum voltage V_{min} on the line
- **c.** The maximum current I_{max} and minimum current I_{min} on the line
- **d.** The power transmitted by the line

3-3. A lossless line has a characteristic impedance of 75 Ω and is terminated in a load of

300 Ω. The line is energized by a generator which has an open-circuit output voltage of 20 V (rms) and output impedance of 75 Ω. The line is assumed to be $2\frac{1}{4}$ wavelengths long.

a. Find the sending-end impedance.
b. Determine the magnitude of the receiving-end voltage.
c. Calculate the receiving-end power at the load.

3-4. A lossless transmission line has a characteristic impedance of 100 Ω and is terminated in a load of 75 Ω. The line is 0.75 wavelength long. Determine:

a. The sending-end impedance
b. The reactance which, if connected across the sending end of the line, will make the input impedance a pure resistance

3-5. A coaxial line with a solid polyethylene dielectric is to be used at a frequency of 3 GHz. Its characteristic impedance Z_0 is 50 Ω and its attenuation constant α is 0.0156 Np/m. The velocity factor which is defined as the ratio of phase velocity over the velocity of light in free space is 0.660. The line is 100 m long and is terminated in its characteristic impedance. A generator, which has an open-circuit voltage of 50 V (rms) and an internal impedance of 50 Ω, is connected to the sending end of the line. The frequency is tuned at 3 GHz. Compute:

a. The magnitude of the sending-end voltage and of the receiving-end voltage
b. The sending-end power and the receiving-end power
c. The wavelengths of the line

3-6. An open-wire transmission line has $R = 5$ Ω/m, $L = 5.2 \times 10^{-8}$H/m, $G = 6.2 \times 10^{-3}$mho/m, and $C = 2.13 \times 10^{-10}$F/m. The signal frequency is 4 GHz. Calculate:

a. The characteristic impedance of the line in both rectangular form and polar form
b. The propagation constant of the wave along the line
c. The normalized impedance of a load $100 + j100$
d. The reflection coefficient at the load
e. The sending-end impedance if the line is assumed a quarter-wavelength long

3-7. A quarter-wave lossless line has a characteristic impedance of 50 Ω and is terminated in a load of 100 Ω. The line is energized by a generator of 20 V (rms) with an internal resistance of 50 Ω. Calculate:

a. The sending-end impedance
b. The magnitude of the receiving-end voltage
c. The power delivered to the load.

3-8. A lossless transmission line is terminated in an open circuit. The sending end is energized by a generator which has an open-circuit output voltage of V_g (rms) and an interval impedance equal to the characteristic impedance of the line. Show that the sending-end voltage is equal to the output voltage of the generator.

3-9. A lossless transmission line has a characteristic impedance of 300 Ω and is operated at a frequency of 10 GHz. The observed standing-wave ratio on the line is 5.0. It is proposed to use a short-circuited stub to match a pure resistor load to the line.

a. Determine the distance in centimeters from the load to the place where the stub should be located. (Two possible solutions.)
b. Find the length of the stub in centimeters. (Two possible solutions.)

3-10. A lossless line has a characteristic impedance of 50 Ω and is loaded by $60 - j60$ Ω. One stub is at the load, and the other is $3\lambda/8$ distance away from the first.

a. Determine the lengths in wavelength of the two short-circuited stubs when a match is achieved.

b. Locate and crosshatch the *forbidden region* of the normalized admittance for possible match.

3-11. A lossless transmission line has a characteristic impedance of 300 Ω and is terminated by an impedance Z_ℓ. The observed standing-wave ratio on the line is 6, and the distance of the first voltage minimum from the load is 0.166λ.

a. Determine the load Z_ℓ.

b. Find the lengths in λ of two shorted stubs, one at the load and one at $\lambda/4$ from the load, which are required to match the load to the line.

3-12. A single-stub tuner is to match a lossless line of 400 Ω to a load of $800 - j300$ Ω. The frequency is 3 GHz.

a. Find the distance in meters from the load to the tuning stub.

b. Determine the length in meters of the short-circuited stub.

3-13. A half-wave-dipole antenna has a driving-point impedance of $73 + j42.5$ Ω. A lossless transmission line connected to a TV set has a characteristic impedance of 300 Ω. The problem is to design a shorted stub with the same characteristic impedance to match the antenna to the line. The stub may be placed at a location closest to the antenna. The reception is assumed to be Channel 83 at a frequency of 0.88525 GHz.

a. Determine the susceptance contributed by the stub.

b. Calculate the length in centimeters of the stub.

c. Find the distance in centimeters between the antenna and the point where the stub is placed. [*Note:* There are two sets of solutions.]

3-14. A lossless transmission line has a characteristic impedance of 100 Ω and is loaded by $100 + j100$ Ω. A single shorted stub with the same characteristic impedance is inserted at $\lambda/4$ from the load to match the line. The load current is measured to be 2 A. The length of the stub is $\lambda/8$.

a. Determine the magnitude and the phase of the voltage across the stub location.

b. Find the magnitude and the phase of the current flowing through the end of the stub.

3-15. A double-stub matching line is shown in Fig. P3-15. The characteristic resistances of the lossless line and stubs are 100 Ω, respectively. The spacing between the two stubs is $\lambda/8$. The load is $100 + j100$. One stub is located at the load. Determine:

a. The reactances contributed by the stub

b. The lengths of the two shorted double-stub tuners [*Note:* There are two sets of solutions.]

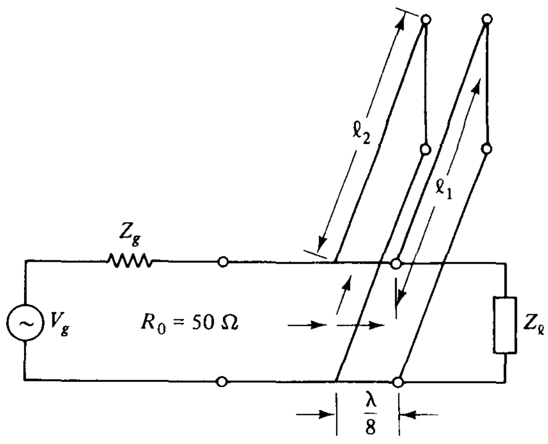

Figure P3-15

3-16. A lossless transmission line has a characteristic impedance Z_0 of 100 Ω and is loaded by an unknown impedance. Its voltage standing-wave ratio is 4 and the first voltage maximum is $\lambda/8$ from the load.

a. Find the load impedance.

b. To match the load to the line, a quarter section of a different line with a characteristic impedance $Z_{01} < Z_0$ is to be inserted somewhere between (in cascade with) the load and the original line. Determine the minimum distance between the load and matching section, and calculate the characteristic impedance Z_{01} in terms of Z_0.

3-17. A lossless transmission line has a characteristic impedance of 100 Ω and is loaded by an unknown impedance. The standing-wave ratio along the line is 2. The first two voltage minima are located at $z = -10$ and -35 cm from the load where $z = 0$. Determine the load impedance.

3-18. A matched transmission line is shown in Fig. P3-18.

a. Find ℓ_1 and d which provide a proper match.

b. With the line and load properly matched determine the VSWR on the section of line between the stubs.

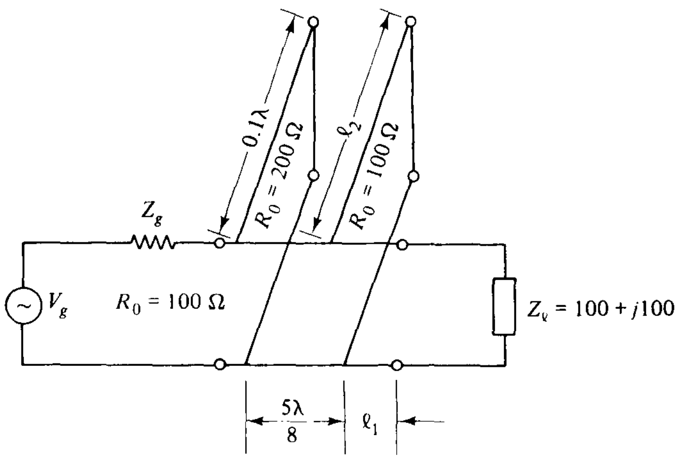

Figure P3-18